

Resonance Transport and Kinetic Entropy

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Abstract

Within the real-time formulation of non-equilibrium field theory generalized transport equations are derived avoiding the standard quasiparticle approximation. They permit to include unstable particles into the transport scheme. In order to achieve a self-consistent, conserving and thermodynamically consistent description, we generalize the Baym's Φ -functional method to genuine non-equilibrium processes. This scheme may be closed at any desired loop order of the diagrams of the functional Φ this way defining a consistent effective theory. By means of a first-order gradient approximation the corresponding Kadanoff-Baym equations are converted into a set of coupled equations. This set consists of a time-irreversible generalized kinetic equation for the slowly varying space-time part of the phase-space distributions and a retarded equation, which provides the fast micro-scale dynamics represented by the four-momentum part of the distributions. Thereby, no constraint to the mass shell of the particles is required any further and the corresponding spectral mass distributions are treated dynamically. The description naturally includes all those quantum features already inherent in the corresponding equilibrium limit (Mat-subara formalism). Memory effects appearing in collision term diagrams of higher order are discussed. The variational properties of Φ -functional permit to derive a generalized expression for the non-equilibrium kinetic entropy flow, which includes corrections from fluctuations and mass width effects. In special cases an H -theorem can be demonstrated implying that the entropy can only increase with time. Memory effects in the kinetic terms provide corrections to the kinetic entropy flow that in equilibrium limit recover the famous bosonic type $T^3 \ln T$ correction to the specific heat of Fermi liquids like Helium-3.

1 Introduction

A proper dynamical scheme in terms of a transport concept that deals with unstable particles, like resonances, i.e. particles with a broad mass width, is still lacking. Rather ad-hoc recipes are in use that sometimes violate basic requirements as given by fundamental

symmetries and conservation laws, detailed balance or thermodynamic consistency. The problem of conserving approximation has first been addressed by Baym and Kadanoff [1,2]. They started from an equilibrium description in the imaginary time formalism and discussed the response to external disturbances. Baym, in particular, showed [2] that any approximation, in order to be conserving, must be so-called Φ -derivable. It turned out that the Φ functional required is precisely the auxiliary functional introduced by Luttinger and Ward [3] (see also ref. [4]) in connection with the thermodynamic potential. In nonequilibrium case the problem of conserving approximations is even more severe than in near-equilibrium linear-response theory.

The appropriate frame for the description of nonequilibrium processes is the real-time formalism of quantum field theory, developed by Schwinger, Kadanoff, Baym and Keldysh [5–7]. In ref. [8] we have introduced the generating functional Φ on the real-time contour. It is determined by the sum of all closed vacuum skeleton diagrams in terms of full classical fields and full Green's functions. All important quantities of a system (such as the sources of classical fields, self-energies, interaction energy, etc.) are derived by variations of the Φ functional with respect to its arguments. The advantage of the Φ functional is that one may formulate various approximations in terms of approximate Φ (so called Φ -derivable approximations), which preserve the conservation laws related to global symmetries of the underlying theory and thermodynamic consistency. Thereby, one may restrict oneself to either few diagrams only or to some sub-set of diagrams for Φ . Note that far not every approximation scheme, e.g., at the level of self-energies, possesses such properties.

Functional variational methods allow to derive the corresponding Dyson equations on real-time contour. The steps towards generalized kinetic equations are provided by the Wigner transformation of all two-point functions and by the first-order gradient approximation. The former formulates all quantities in terms of phase-space distribution functions in four dimensions, i.e. as a function of energy and momentum for any space-time coordinate. The gradient approximation leads to Poisson-bracket expressions which permit a classical interpretation. For complicated collision terms, the latter also give rise to memory effects. The quasiparticle approximation is not required at any step. Rather one can keep the finite damping width of the particles. The corresponding dynamical information contained in the spectral functions is determined by local retarded equations. We derive closed expressions for the nonequilibrium energy momentum tensor in the Φ -derivable scheme, which also include terms arising from the fluctuations in the system and the finite mass width of the particles.

The second main issue of this paper is the role of entropy. Although the entropy is a central quantity in thermodynamics and statistical mechanics, many problems connected with it, in particular, its description in terms of Green's functions in the nonequilibrium case, are still open. One can find related discussions in many textbooks and reviews, e.g., in refs [9–13]. The thermodynamic entropy has extensively been discussed in the literature at the end of sixties and beginning of seventies, cf. refs [14–16] and references therein. The fact that the generalized kinetic equation possesses a proper thermodynamic limit, does not yet imply that this limit will be approached during the evolution. The latter is however

ensured, if one can prove an H -theorem for the equations of motion. Using the Φ -derivable properties we are able to get an expression, which takes the sense of a nonequilibrium kinetic entropy expressed in terms of Green's functions and self-energies. It generalizes to nonequilibrium the well known equilibrium expression obtained in Matsubara formalism within the Φ -functional scheme. We derived conditions for an H -theorem, which apply in many cases discussed in transport problems. In addition, the kinetic entropy expression is extended beyond the quasiparticle limit thus accounting for the fluctuations. Memory effects lead to further corrections to the entropy flow. The latter have to be established for each particular case. In spite of many attempts so far, H -theorem has not yet been proven even for the classical kinetic equation including triple collision term, cf. ref. [13]. Therefore, up to now there exists no appropriate kinetic entropy expression (derived from generalized kinetic description). Our demonstration of H -theorem for particular cases including the fluctuation parts may be viewed as step towards this general goal.

The paper is organized as follows. Within the real-time contour formulation of nonequilibrium dynamics we define a generating functional Γ together with the auxiliary Φ functional of Baym, cf. ref. [8], (sect. 2). Then, for slowly varying generalized distribution functions we derive a generalized kinetic equation consistent with Φ , using the gradient approximation (sect. 3). Arguments are given that the generalized distribution functions remain positive during the time evolution and thus permit a probabilistic interpretation in four-momentum space at each space-time point. The properties of Φ determine the conservation laws in terms of conserved Noether currents and energy-momentum tensor. Further properties are exploited with the help of the decomposition rules for the diagrams of Φ formulated in terms of full Green's functions [17]. These rules lead to a multi-process decomposition of a Φ -derivable collision term, which contains space-time local and non-local (memory) parts (sect. 4). Time-irreversibility is discussed in connection with our generalized kinetic description, and expression for the kinetic entropy flow is derived. In specific cases an explicit growth of the kinetic entropy with time, i.e. an H -theorem (sect. 5) is demonstrated. The proof once again relies on the Φ -derivability of the collision term. Markovian and memory contributions to the entropy flow are discussed at a specific example. Finally, the proper thermodynamic limit of the kinetic entropy is demonstrated (sect. 6). Some formal details are deferred to Appendices.

To be specific, we concentrate on systems of nonrelativistic particles. Bosonic mean fields are not treated in this paper; they can however be included along the lines given in ref. [8]. Whenever possible we omit the index a denoting the different particle species and intrinsic quantum numbers in a multi-component system.

2 Generating Functionals Γ and Φ

2.1 Φ Functional and Dyson's Equation on Real-Time Contour

We assume the nonequilibrium system to be prepared at some initial time t_0 in terms of a given density operator $\hat{\rho}_0 = \sum_{\alpha} P_{\alpha} |\alpha\rangle \langle \alpha|$, where the $|\alpha\rangle$ form a complete set of eigenstates of $\hat{\rho}_0$.

All observables can be expressed through n -point Wightman functions of Heisenberg operators $\hat{A}(t_1), \dots, \hat{O}(t_n)$ at some later times

$$\begin{aligned} \langle \hat{O}(t_n) \dots \hat{B}(t_2) \hat{A}(t_1) \rangle &=: \text{Tr } \hat{O}(t_n) \dots \hat{B}(t_2) \hat{A}(t_1) \hat{\rho}_0(t_0) \\ &= \sum_{\alpha} P_{\alpha} \langle \alpha | \hat{O}(t_n) \dots \hat{B}(t_2) \hat{A}(t_1) | \alpha \rangle. \end{aligned} \quad (2.1)$$

Note the fixed operator ordering for Wightman functions.

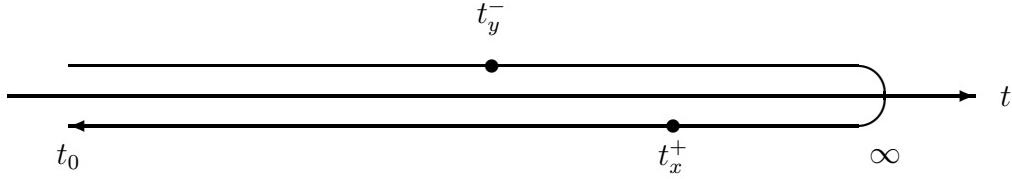


Figure 1: Closed real-time contour with two external points x, y on the contour.

The nonequilibrium theory can entirely be formulated on a *single* special contour, the *closed real-time contour* (see figure 1) with the time argument running from t_0 to ∞ along *time-ordered* branch and back to t_0 along *anti-time-ordered* branch. The contour Green's function is defined as the expectation value of contour-ordered products of operators

$$iG(x, y) = \langle \mathcal{T}_{\mathcal{C}} \hat{\varphi}(x) \hat{\varphi}^\dagger(y) \rangle, \quad (2.2)$$

where $\mathcal{T}_{\mathcal{C}}$ denotes the special time-ordering operator, which orders the operators according to a time parameter running along the time contour \mathcal{C} . With the aim to come to a self-consistent treatment in terms of two-point Green's functions, one has to ignore higher order correlations which would be present in the exact dynamical equation for two-point Green's functions. This step is achieved by the Wick decomposition which leads to the Dyson equation now formulated on the real time contour

$$S_x G(x, y) = \delta_{\mathcal{C}}(x, y) + \int_{\mathcal{C}} dz \Sigma(x, z) G(z, y), \quad (2.3)$$

$$S_x G^0(x, y) = \delta_C(x, y), \quad (2.4)$$

where

$$S_x = i\partial_t + \frac{1}{2m} \partial_x^2 \quad (2.5)$$

in nonrelativistic kinematics. Here $\delta_C(x, y)$ is δ -function on the contour, G^0 is the free Green's function and Σ is the self-energy, expressed via one-particle Green's function in the usual way. This way one ignores higher order correlation functions, cf. refs [18,19]. Then the self-energy itself becomes a functional of the Green's functions (2.2) with the consequence that the standard Wick decomposition can be applied for the self-energy and one thus obtains standard contour diagrams, cf. ref. [8]. In such a diagrammatic representation eq. (2.3) is given by

(2.6)

with the two-point functions $-i\Sigma\{G(x, y)\}$, as driving terms, and the Green's functions

$$iG^0(x, y) = \overline{x \leftarrow y}, \quad iG(x, y) = \overline{x \leftarrow y}. \quad (2.7)$$

The arrow always points towards the $\hat{\varphi}$ -field operator in the contour ordered expression, cf. eq. (2.2).

The physical condition, which justifies to disregard higher order correlations, is that the typical interaction time τ_{int} for the change of the correlation functions is much less than the typical relaxation time τ_{rel} , which determines the system evolution. Describing the system at times $t - t_0 \gg \tau_{\text{int}}$, one can neglect initial correlations, since they are supposed to be dying out at times $\sim \tau_{\text{int}}$ in accordance with the Bogolyubov's principle of the weakening of initial correlations.¹ This coarse-graining leads to time-irreversibility, as we will see below. Alternatively, one also could suppose the initial state to be uncorrelated, like an equilibrium ideal gas state, which corresponds to an information loss right from the beginning, cf. refs [6,20].

In the imaginary-time formalism, Luttinger and Ward [3] have shown that the generating functional Γ can be expressed in terms of an auxiliary functional Φ , where Φ is solely given in terms of full, i.e. resummed, propagators. Following ref. [8], it can be generalized to the real-time case with the following diagrammatic representation

¹ Actually, considering dilute systems, Bogolyubov suggested the weakening of all correlations, whereas we use a weaker assumption on the weakening of only short-time ($\sim \tau_{\text{int}}$) correlations, cf. ref. [12].

$$\begin{aligned}
& i\Gamma \{G, \lambda\} = i\Gamma^0 \{G^0\} \\
& + \left\{ \sum_{n_\Sigma} \frac{1}{n_\Sigma} \underbrace{\text{Diagram with } n_\Sigma \text{ self-energy insertions}}_{\pm \ln(1 - \odot G^0 \odot \Sigma)} - \underbrace{\text{Diagram with } n_\lambda \text{ vertices}}_{\pm \odot G \odot \Sigma} \right\} + \sum_{n_\lambda} \frac{1}{n_\lambda} \underbrace{\text{Diagram with } n_\lambda \text{ vertices}}_{+ i\Phi \{G, \lambda\}} . \tag{2.8}
\end{aligned}$$

Here upper signs relate to fermion quantities, whereas lower signs, to boson ones. For the sake of mathematical convenience, cf. ref. [8], a scaling parameter λ is introduced in each interaction vertex. For the physical case, $\lambda = 1$. The integer n_Σ counts the number of self-energy insertions in the ring diagrams. For the closed diagrams of Φ the value n_λ counts the number of vertices building up each such diagram. Due to the global factor $1/n_\lambda$ the set of Φ -diagrams is not resumable in the standard diagrammatic sense. For nonrelativistic instantaneous two-body interaction each interaction counts as two vertices, cf. Appendix C. The term Γ^0 solely depends on the free propagator G^0 . The diagrams contributing to Φ are given in terms of full propagators G (thick lines). These diagrams are *two-particle irreducible* (label *c2*), i.e. they cannot be decomposed into two pieces by cutting two propagator lines. The latter property matches diagrammatic rules for the resummed self-energy $\Sigma(x, y)$, which results from functional variation of Φ with respect to the propagator $G(y, x)$, i.e.

$$-i\Sigma = \mp \delta i\Phi / \delta iG. \tag{2.9}$$

Relation (2.9) directly follows from the stationarity condition of Γ (2.8) with respect to variations in the full propagator G on the contour

$$\delta\Gamma \{G, \lambda\} / \delta G = 0, \tag{2.10}$$

which also provides the Dyson's equation (2.3). In graphical terms, the variation (2.9) is realized by opening a propagator line in any diagram of Φ . The resulting set of thus opened diagrams must then be that of proper skeleton diagrams of Σ in terms of *full propagators*, i.e. void of any self-energy insertions.

Subsequently we will not use the contour representation but rather its decomposition into the two branches the time-ordered and the anti-time ordered branch using a $\{-+\}$ matrix notation as explained in Appendix A, cf. also refs [21,8].

2.2 Φ -Derivable Approximation Scheme

For any practical calculation one has to truncate the scheme. In the weak-coupling limit, the perturbative expansion may be restricted to a certain order. Then no particular problems are encountered as far as conservation laws are concerned, since they are fulfilled order by order in perturbation theory. On the other hand, such perturbative expansion may not be adequate, as, for example, in the strong coupling limit, where resummation concepts have to be applied. Such schemes sum up certain sub-series of diagrams to any order. Furthermore, with the aim to solve dynamical equations of motion, such as transport equations, one automatically resums all terms in the equations of motion to any order. For such resumming schemes, the situation with conservation laws is not quite as clear.

We consider the so-called Φ -derivable approximations, first introduced by Baym [2] within the imaginary time formulation. A Φ -derivable approximation is constructed by confining the infinite set of diagrams for Φ to either only a few of them or some sub-series of them. Note that Φ itself is constructed in terms of “full” Green’s functions, where “full” now takes the sense of solving self-consistently the Dyson’s equation with the driving terms derived from this Φ through relation (2.9). It means that even restricting ourselves to a single diagram in Φ , in fact, we deal with a whole sub-series of diagrams in terms of free Green’s functions, and “full” takes the sense of the sum of this whole sub-series. Thus, a Φ -derivable approximation offers a natural way of introducing closed, i.e. consistent approximation schemes based on summation of diagrammatic sub-series. In order to preserve the original symmetry of the exact Φ , we postulate that the set of diagrams defining the Φ -derivable approximation complies with all such symmetries. As a consequence, approximate forms of $\Phi^{(\text{appr.})}$ define *effective theories*, where $\Phi^{(\text{appr.})}$ serves as a generating functional for approximate self-energies $\Sigma^{(\text{appr.})}(x, y)$

$$-\text{i}\Sigma^{(\text{appr.})}(x, y) = \mp \frac{\delta\text{i}\Phi^{(\text{appr.})}}{\delta\text{i}G^{(\text{appr.})}(y, x)} \quad (2.11)$$

which then enter as driving terms for the Dyson’s equations (2.3). The propagators, solving this set of Dyson’s equations, are still called “full” in the sense of the $\Phi^{(\text{appr.})}$ -derivable scheme. Below, we omit the superscript “appr.”.

3 Generalized Kinetic Equation

3.1 Gradient Expansion Scheme

For slightly inhomogeneous and slowly evolving systems, the degrees of freedom can be subdivided into rapid and slow ones. Any kinetic approximation is essentially based on

this assumption. Then for any two-point function $F(x, y)$, one separates the variable $\xi = (t_1 - t_2, \mathbf{r}_1 - \mathbf{r}_2)$, which relates to rapid and short-ranged microscopic processes, and the variable $X = \frac{1}{2}(t_1 + t_2, \mathbf{r}_1 + \mathbf{r}_2)$, which refers to slow and long-ranged collective motions. The Wigner transformation, i.e. the Fourier transformation in four-space difference $\xi = x - y$ to four-momentum p leads to the corresponding Wigner densities in four-phase-space. Since the Wigner transformation is defined for physical space-time coordinates rather than for contour coordinates one has to decompose the contour integrations into its two branches, the time-ordered $\{-\}$ branch and the anti-time ordered $\{+\}$ branch. This is explained in detail in Appendix A. Two-point functions then become matrices of the contour decomposed $\{-+\}$ components with physical space-time arguments. Thus

$$F^{ij}(X; p) = \int d\xi e^{ip\xi} F^{ij}(X + \xi/2, X - \xi/2), \quad i, j \in \{-+\} \quad (3.1)$$

leads to a four-phase-space representation of two-point functions. The retarded or advanced relations between two-point functions formulated on real-time contour and those in matrix notation, F^{ij} , presented in Appendix A, are thereby preserved.

The Wigner transformation of Dyson's equation (2.3) is straight forward. Taking the difference and half-sum of Dyson's equation (2.3) and the corresponding adjoint equation and doing the Wigner transformation we arrive at equations

$$iv_\mu \partial_X^\mu G^{ij}(X, p) = \int_C d\xi e^{ip\xi} \int dz (\Sigma(x^i, z)G(z, y^j) - G(x^i, z)\Sigma(z, y^j)), \quad (3.2)$$

$$\hat{Q}_X G^{ij}(X, p) = \sigma^{ij} + \frac{1}{2} \int_C d\xi e^{ip\xi} \int dz (\Sigma(x^i, z)G(z, y^j) + G(x^i, z)\Sigma(z, y^j)), \quad (3.3)$$

where σ^{ij} accounts for the integration sense on the two contour branches, cf. eqs (A.2), (A.3). For nonrelativistic kinematics $v^\mu = (1, \mathbf{p}/m)$, and $\hat{Q}_X = p_0 - \mathbf{p}^2/2m - \partial_X^2/8m$. In this matrix notation, two of equations (3.2) and (3.3), involving G^{-+} and G^{+-} on the left-hand side, are known as Kadanoff-Baym equations in Wigner representation [6]. Particular combinations of these equations lead to the retarded and advanced equations which completely decouple and involve only integrations over physical times rather than contour times.

Standard transport descriptions usually involve two further approximation steps: (i) the gradient expansion for the slow degrees of freedom, as well as (ii) the quasiparticle approximation for rapid ones. We intend to avoid the latter approximation and will solely deal with the gradient approximation for slow collective motions by performing the gradient expansion of eqs (3.2) and (3.3). This step preserves all the invariances of the Φ functional in a Φ -derivable approximation.

Technically, the gradient expansion is performed as follows. The Wigner transformation of a convolution of two two-point functions is given by

$$\int d\xi e^{ip\xi} \left(\int dz f(x, z) \varphi(z, y) \right) = \left(\exp \left[\frac{i\hbar}{2} (\partial_p \partial_{X'} - \partial_X \partial_{p'}) \right] f(X, p) \varphi(X', p') \right)_{p'=p, X'=X}. \quad (3.4)$$

Here, we have temporarily restored \hbar for the sake of physical lucidity. Expanding this expression in powers of $\hbar \partial_X \cdot \partial_p$ to the first order, we obtain

$$\int d\xi e^{ip\xi} \left(\int dz f(x, z) \varphi(z, y) \right) \simeq f(X, p) \varphi(X, p) + \frac{i\hbar}{2} \{f(X, p), \varphi(X, p)\}, \quad (3.5)$$

where

$$\{f(X, p), \varphi(X, p)\} = \frac{\partial f}{\partial p^\mu} \frac{\partial \varphi}{\partial X_\mu} - \frac{\partial f}{\partial X^\mu} \frac{\partial \varphi}{\partial p_\mu} \quad (3.6)$$

is the classical Poisson bracket in covariant notation. We would like to stress that the smallness of the $\hbar \partial_X \cdot \partial_p$ comes solely from the smallness of space-time gradients ∂_X , while momentum derivatives ∂_p are not assumed to be small! They must be not too large only in order to preserve the smallness of the total parameter $\hbar \partial_X \cdot \partial_p$. This point is sometimes incorrectly treated in the literature.

3.2 Kinetic Equation in Physical Notation

It is helpful to avoid all the imaginary factors inherent in the standard Green's function formulation and introduce two quantities which are real and, in the quasi-homogeneous limit, positive, cf. subsect. 3.5, and therefore have a straightforward physical interpretation, much like for the Boltzmann equation. We define

$$\begin{aligned} F(X, p) &= A(X, p) f(X, p) = (\mp) iG^{-+}(X, p), \\ \tilde{F}(X, p) &= A(X, p) [1 \mp f(X, p)] = iG^{+-}(X, p), \end{aligned} \quad (3.7)$$

for the generalized Wigner functions F and \tilde{F} and the corresponding *four*-phase-space distribution functions $f(X, p)$ and Fermi/Bose factors $[1 \mp f(X, p)]$. Here

$$A(X, p) \equiv -2\text{Im}G^R(X, p) = \tilde{F} \pm F = i(G^{+-} - G^{-+}) \quad (3.8)$$

is the spectral function, where G^R is the retarded propagator, cf. eq. (A.6) of Appendix A. According to relations (A.6) and (A.7) between Green's functions G^{ij} , *only two real functions of all these G^{ij} are required for a complete description of the system's evolution.*

The reduced gain and loss rates of the collision integral are defined as

$$\Gamma_{\text{in}}(X, p) = \Gamma(X, p)\gamma(X, p) = (\mp)i\Sigma^{-+}(X, p), \quad (3.9)$$

$$\Gamma_{\text{out}}(X, p) = \Gamma(X, p)[1 \mp \gamma(X, p)] = i\Sigma^{+-}(X, p) \quad (3.10)$$

with the damping width

$$\Gamma(X, p) \equiv -2\text{Im}\Sigma^R(X, p) = \Gamma_{\text{out}}(X, p) \pm \Gamma_{\text{in}}(X, p), \quad (3.11)$$

where Σ^R is the retarded self-energy, cf. eq. (A.6). The opposite combination

$$I(X, p) = [2\gamma \mp 1]\Gamma = \Gamma_{\text{in}}(X, p) \mp \Gamma_{\text{out}}(X, p), \quad (3.12)$$

is related to fluctuations. The dimensionless quantity γ is introduced for further convenience.

In terms of the new notation (3.7)–(3.11) and within the first-order gradient approximation, the equations (3.2) for F and \tilde{F} take the form

$$\mathcal{D}F(X, p) - \{\Gamma_{\text{in}}, \text{Re}G^R\} = C(X, p), \quad (3.13)$$

$$\mathcal{D}\tilde{F}(X, p) - \{\Gamma_{\text{out}}, \text{Re}G^R\} = \mp C(X, p) \quad (3.14)$$

which we denote as the transport equations. Here the differential drift operator is defined as

$$\mathcal{D} = \left(v_\mu - \frac{\partial \text{Re}\Sigma^R}{\partial p^\mu} \right) \partial_X^\mu + \frac{\partial \text{Re}\Sigma^R}{\partial X^\mu} \frac{\partial}{\partial p_\mu}, \quad v^\mu = (1, \mathbf{p}/m). \quad (3.15)$$

Furthermore

$$C(X, p) = \Gamma_{\text{in}}(X, p)\tilde{F}(X, p) - \Gamma_{\text{out}}(X, p)F(X, p) = A\Gamma[\gamma - f] \quad (3.16)$$

is the collision term with the dimensionless functions f and γ defined in (3.7) and (3.9), while $\{\Gamma_{\text{in}}, \text{Re}G^R\}$ and $\{\Gamma_{\text{out}}, \text{Re}G^R\}$ are the fluctuation terms. Within the same approximation level eq. (3.3) provides us with two alternative equations for F and \tilde{F}

$$MF - \text{Re}G^R\Gamma_{\text{in}} = \frac{1}{4} (\{\Gamma, F\} - \{\Gamma_{\text{in}}, A\}), \quad (3.17)$$

$$M\tilde{F} - \text{Re}G^R\Gamma_{\text{out}} = \frac{1}{4} (\{\Gamma, \tilde{F}\} - \{\Gamma_{\text{out}}, A\}) \quad (3.18)$$

with the "mass" function

$$M(X, p) = p_0 - \frac{1}{2m} \mathbf{p}^2 - \text{Re}\Sigma^R(X, p), \quad (3.19)$$

which relates to the drift operator via $\mathcal{D}f = \{M, f\}$ for any four-phase-space function f . Eqs (3.17), (3.18) are reported as the mass shell equations. Appropriate combinations of the two sets of equations (3.13)–(3.14) and (3.17)–(3.18) provide us with the retarded equations

$$\mathcal{D}G^R(X, p) + \frac{i}{2} \{ \Gamma, G^R \} = 0, \quad (3.20)$$

$$\left(M(X, p) + \frac{i}{2} \Gamma(X, p) \right) G^R(X, p) = 1. \quad (3.21)$$

This subset (3.20)–(3.21) is solved by [22]

$$G^R = \frac{1}{M(X, p) + i\Gamma(X, p)/2} \Rightarrow \begin{cases} A(X, p) = \frac{\Gamma(X, p)}{M^2(X, p) + \Gamma^2(X, p)/4}, \\ \text{Re}G^R(X, p) = \frac{M(X, p)}{M^2(X, p) + \Gamma^2(X, p)/4}. \end{cases} \quad (3.22)$$

The spectral function satisfies the sum-rule

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} A(X, p) = 1, \quad (3.23)$$

which follows from the canonical equal-time (anti) commutation relations for (fermionic) bosonic field operators.

With the solution (3.22) for G^R equations (3.13) and (3.17) become identical to (3.14) and (3.18), respectively. However, equations (3.13) and (3.17) are nonidentical, while they were identical before the gradient expansion. We will now discuss under which conditions they are equivalent. First, this takes place in the standard quasiparticle limit when one puts $\Gamma \rightarrow 0$ in the Green's functions. This limit will be discussed in the next subsect. The equivalence is also given in the case we are interested in here, where Γ is of finite value, however, one needs $|f - \gamma| \ll 1$ which does not necessarily require the system to be close to local equilibrium. In both cases the gradient approximation implies that the macroscopic time scale τ_{macro} , characterizing kinetic processes, is much larger than the microscopic time scale τ_{micro} , relating to rapid microscopic processes.

Using the limit $|f(X, p) - \gamma(X, p)| \ll 1$ and writing all $\Gamma_{in} = \gamma\Gamma$ terms as $(f + (\gamma - f))\Gamma$, the transport equation (3.13) and the mass-shell equation (3.17) take the forms

$$\frac{A^2}{2} (\Gamma \{M, f\} - M \{\Gamma, f\}) = \Gamma A(\gamma - f) - r_{\text{kin}}, \quad (3.24)$$

$$\frac{A \text{Re}G^R}{2} (\Gamma \{M, f\} - M \{\Gamma, f\}) = MA(\gamma - f) - r_{\text{mass-eq}}, \quad (3.25)$$

where the remaining terms r_{kin} and $r_{\text{mass-shell}}$ are of order $(\gamma - f)$ times gradient terms, i.e. of second order in the gradient expansion². From eq. (3.22) one sees that $\Gamma \text{Re}G^R = MA$. Using this we observe that the mass-shell equation loses its original sense, since to leading order relation (3.25) is equivalent to the kinetic equation (3.24). However, the still remaining difference in the second-order terms is inconvenient from the practical point of view. Following Botermans and Malfliet [22] who first suggested to drop the above r_{kin} term in the kinetic equation we put $r_{\text{kin}} = r_{\text{mass}} = 0$, since then both equations indeed become identical. This step amounts to replace the Γ_{in} and Γ_{out} terms by $f\Gamma$ and $(1 \mp f)\Gamma$ in *all Poisson brackets*. The so obtained *generalized kinetic equations* for F and \tilde{F} then read

$$\mathcal{D}F(X, p) - \left\{ \Gamma \frac{F}{A}, \text{Re}G^R \right\} = C(X, p), \quad (3.27)$$

$$\mathcal{D}\tilde{F}(X, p) - \left\{ \Gamma \frac{\tilde{F}}{A}, \text{Re}G^R \right\} = \mp C(X, p). \quad (3.28)$$

Although these two equations are identical to each other, we nevertheless have presented both of them for the sake of further convenience. With the help of the retarded relations (3.22) both equations can be converted to one generalized transport equation for the phase-space occupation functions f

$$\frac{A^2}{2} (\Gamma \mathcal{D}f - M \{\Gamma, f\}) = C. \quad (3.29)$$

To get eq. (3.29) we used that the additional Poisson-bracket fluctuation term becomes

$$\left\{ \Gamma f, \text{Re}G^R \right\} = \frac{M^2 - \Gamma^2/4}{(M^2 + \Gamma^2/4)^2} \mathcal{D}(\Gamma f) + \frac{M\Gamma^2/2}{(M^2 + \Gamma^2/4)^2} \{\Gamma, f\}. \quad (3.30)$$

We now provide a physical interpretation of various terms in the generalized kinetic equation (3.27) or (3.29). This physical interpretation relies on the similarity of most of the terms to conventional kinetic equations, for example, such as the Landau equation for

² These terms have the explicit form

$$r_{\text{kin}} = -\{\Gamma(\gamma - f), \text{Re}G^R\}, \quad r_{\text{mass-eq}} = \frac{1}{4} \{\Gamma(\gamma - f), A\}. \quad (3.26)$$

Fermi liquids (see, e.g., refs [16,21]) and ref. [23] for the relativistic version), proposed by Landau on the basis of an intuitive quasiparticle picture [24].

For this purpose it is advantage to convert the drift operator (3.15) into a space and time separated form

$$\mathcal{D} = \frac{1}{Z} (\partial_t + \mathbf{v}_g \partial_{\mathbf{X}}) + \partial_t \text{Re}\Sigma^R \cdot \partial_{p_0} - \partial_{\mathbf{X}} \text{Re}\Sigma^R \cdot \partial_{\mathbf{p}}, \quad (3.31)$$

where

$$\mathbf{v}_g(X, p) = Z \left(\mathbf{v} + \frac{\partial \text{Re}\Sigma^R}{\partial \mathbf{p}} \right) \quad \text{with} \quad Z = \left(v_0 - \frac{\partial \text{Re}\Sigma^R}{\partial p_0} \right)^{-1} \quad (3.32)$$

takes the meaning of the group 3-velocity in the quasiparticle approximation, while Z is the standard renormalization factor.

Thus, the drift term $\mathcal{D}F$ on the l.h.s. of eq. (3.27) is the usual kinetic drift term including the corrections from the self-consistent field $\text{Re}\Sigma^R$ into the convective transfer of real and also virtual particles. In the collisionless case $\mathcal{D}F = 0$ (Vlasov equation), the quasi-linear first-order differential operator \mathcal{D} defines characteristic curves. They are the standard classical paths. The four-phase-space probability $F(X, p)$ is conserved along these paths. The formulation in terms of a Poisson bracket in four dimensions implies a generalized Liouville theorem. In the collisional case, both the collision, C , and fluctuation terms (3.30) change the phase-space probabilities of the “generalized” particles during their propagation along the “generalized” classical paths given by \mathcal{D} . We use the term “generalized” in order to emphasize that particles are no longer bound to their mass-shell, $M = 0$, during propagation due to the collision term, i.e. due to decay, creation or scattering processes.

The r.h.s. of eq. (3.27) specifies the collision term C in terms of gain and loss terms, which also can account for multi-particle processes. Since F includes a factor A , C further deviates from the standard Boltzmann-type form in as much that it is multiplied by the spectral function A , which accounts for the finite width of the particles. The Poisson-bracket term (3.30) is special. It contains genuine contributions from the finite mass width of the particles and describes the response of the surrounding matter due to fluctuations. This can be seen from the conservation laws discussed below. In particular the first term in (3.30) gives rise to a back-flow component of the surrounding matter. It restores the Noether currents as the conserved ones rather than the intuitively expected sum of convective currents arising from the convective $\mathcal{D}F$ terms in (3.27). In eq. (3.29) one further realizes that the function $\frac{1}{2}A^2\Gamma$ in front of the drift term is more sharply peaked than the original spectral function A . Both $\frac{1}{2}A^2\Gamma$ and A are reduced to the same δ -function in the quasiparticle limit. The $M\{\Gamma, f\}$ term, which gives no contribution in the quasiparticle limit due to the factor M , represents a specific off-mass-shell response, cf. ref. [25].

So far the gradient approximation has been applied to the space-time foldings occurring

between the self-energies and the propagators appearing in the collision term. They give rise to the gradient terms on the l.h.s. of the kinetic equations. This is sufficient as long as the self-energies are calculated without further approximation. Commonly one likes to obtain also Γ_{in} and Γ_{out} in a kind of local approximation evaluated with all Green's functions taken at the same space-time point X . Then for self-energy diagrams with more than two points, also nonlocal gradient corrections within these diagrams arise, which have to be accounted for in a consistent gradient approximation scheme. The latter fact gives rise to memory effects, which will be discussed in detail in subsect. 4.2, below.

In all derivations above the Γ_{in} and Γ_{out} terms in all Poisson brackets have consistently been replaced by $f\Gamma$ and $(1 \mp f)\Gamma$, respectively. In fact, this procedure is optional. However, in doing so we gained few advantages. First, all our generalized kinetic equations, (3.24), and mass-shell equations, (3.25) are *exactly* equivalent to each other, as they were in exact theory before the gradient expansion. Second, the so obtained generalized kinetic equation (3.29) has particular feature with respect to the definition of a nonequilibrium entropy flow in connection with the formulation of an *exact* H-theorem in certain cases (sect. 5). If we refrained from these substitutions, these both features would become approximate with the accuracy up to second-order gradients. While this accuracy still complies with the gradient approximation level, it makes our first-order approximation less elegant.

3.3 Quasiparticle Limit

The quasiparticle picture is recovered in the limit of small (while nonzero) width Γ . Small means that Γ should be less than all typical scales in the problem. This allows to put Γ zero in all Green's functions which are, thereby, called quasiparticle Green's functions. nonzero means that Γ is kept finite in the collision term being now calculated with the help of these quasiparticle Green's functions.

Thus, in this limit the spectral function (3.22) takes the form

$$A^{\text{qp}} = 2\pi \text{ sign}\Gamma \delta(M) \quad (3.33)$$

with M given by eq. (3.19). As the right hand side of the mass shell equation (3.17) contains one Γ factor more than the left hand side, it reduces to $MA(f - \gamma) = 0$ requiring M to vanish within the range of the spectral function. This just complies with the on-shell δ -function in eq. (3.33) which with $M = 0$ provides the dispersion equation for the quasiparticle energies $\varepsilon(X, \mathbf{p})$

$$\varepsilon(X, \mathbf{p}) = \frac{1}{2m} \mathbf{p}^2 + \text{Re } \Sigma^R(X, \varepsilon(X, \mathbf{p}), \mathbf{p}). \quad (3.34)$$

This dispersion equation may have several solutions (branches). For brevity we drop a possible branch index. In the quasiparticle limit the generalized Wigner functions and the

Fermi/Bose factors (3.7) are presented in the form

$$F^{\text{qp}}(X, p) = 2\pi Z^{\text{qp}}(X, \mathbf{p}) \delta(p_0 - \varepsilon(X, \mathbf{p})) f^{\text{qp}}(X, \mathbf{p}), \quad (3.35)$$

$$\tilde{F}^{\text{qp}}(X, p) = 2\pi Z^{\text{qp}}(X, \mathbf{p}) \delta(p_0 - \varepsilon(X, \mathbf{p})) [1 \mp f^{\text{qp}}(X, \mathbf{p})], \quad (3.36)$$

with the quasiparticle distribution functions

$$f^{\text{qp}}(X, \mathbf{p}) = f(X, \varepsilon(X, \mathbf{p}), \mathbf{p}), \quad (3.37)$$

which depend on the 3-momentum only rather than on the 4-momentum, while

$$Z^{\text{qp}}(X, \mathbf{p}) = Z(X, \varepsilon(X, \mathbf{p}), \mathbf{p}) \quad (3.38)$$

denotes the quasiparticle normalization factor (cf. eq. (3.32)). In practice, Z^{qp} is always positive for truly nonrelativistic systems [26]. We have replaced p_0 by $\varepsilon(X, \mathbf{p})$ in all relations above due to the on-shell δ -function in eq. (3.33).

Integration of eq. (3.27) over p_0 in a narrow region of width Γ around the branch position $\varepsilon(X, \mathbf{p})$, where one can identify $A = A^{\text{qp}}$, leads to the standard quasiparticle transport equation

$$\partial_t f^{\text{qp}}(X, \mathbf{p}) + \frac{\partial \varepsilon(X, \mathbf{p})}{\partial \mathbf{p}} \partial_{\mathbf{x}} f^{\text{qp}}(X, \mathbf{p}) - \partial_{\mathbf{x}} \varepsilon(X, \mathbf{p}) \frac{\partial f^{\text{qp}}(X, \mathbf{p})}{\partial \mathbf{p}} = C^{\text{qp}}(X, \mathbf{p}). \quad (3.39)$$

Here $C^{\text{qp}}(X, \mathbf{p})$ is the corresponding collision integral now expressed in terms of quasiparticle distributions. The fluctuation term $\{\Gamma f, \text{Re}G^R\}$ integrated over p_0 becomes of second order in Γ and thus drops out of this quasiparticle equation.

This quasiparticle transport equation is consistent as long as its evolution does not lead beyond its region of validity, i.e. into region where Γ becomes comparable to typical energy scales in the problem. For instance, the consistency of the Landau Fermi-liquid theory is preserved, provided only low-excited Fermi systems are considered. Then the Pauli-blocking prevents particles to scatter to states laying far away from the Fermi surface, i.e. to a region where the quasiparticle approximation would be violated.

In above approach neither any uncontrolled approximations nor any specific ansatz has been introduced. The quasiparticle limit of the generalized kinetic equation directly follows from our consideration and encounters no problems for the eliminating the extra Poisson bracket term. Here a caution should be expressed. The quasiparticle form of the spectral function of eq. (3.33) gives

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} A^{\text{qp}}(X, p) = Z^{\text{qp}}(X, \mathbf{p}) \quad (3.40)$$

for the sum-rule rather than unity. The Z^{qp} factor appears here, because in (3.40) we artificially extended the quasiparticle form of A to all energies and did not account for background terms outside the quasiparticle region. However, in this form the sum rule is often taken as the canonical one for quasiparticles, cf. ref. [27], in order to construct an effective theory only in terms of quasiparticles. This recipe works, if the quasiparticle energy region is indeed well separated from the region, where particles have a rather large width.

3.4 Conservations of Charge and Energy–Momentum

Transport equation (3.27) weighted either with the charge e or with 4-momentum p^ν , integrated over momentum and summed over internal degrees of freedom like spin (Tr) gives rise to the charge or energy–momentum conservation laws, respectively, with the Noether 4-current and Noether energy–momentum tensor defined by the following expressions

$$j^\mu(X) = e\text{Tr} \int \frac{d^4p}{(2\pi)^4} v^\mu F(X, p), \quad (3.41)$$

$$\Theta^{\mu\nu}(X) = \text{Tr} \int \frac{d^4p}{(2\pi)^4} v^\mu p^\nu F(X, p) + g^{\mu\nu} (\mathcal{E}^{\text{int}}(X) - \mathcal{E}^{\text{pot}}(X)). \quad (3.42)$$

Here

$$\mathcal{E}^{\text{int}}(X) = \left\langle -\hat{\mathcal{L}}^{\text{int}}(X) \right\rangle = \left. \frac{\delta \Phi}{\delta \lambda(x)} \right|_{\lambda=1} \quad (3.43)$$

is interaction energy density, which in terms of Φ is given by a functional variation with respect to a space-time dependent coupling strength of interaction part of the Lagrangian density $\hat{\mathcal{L}}^{\text{int}} \rightarrow \lambda(x)\hat{\mathcal{L}}^{\text{int}}$, cf. ref. [8]. The potential energy density \mathcal{E}^{pot} takes the form

$$\mathcal{E}^{\text{pot}} = \text{Tr} \int \frac{d^4p}{(2\pi)^4} \left[\text{Re} \Sigma^R F + \text{Re} G^R \frac{\Gamma}{A} F \right]. \quad (3.44)$$

Whereas the first term in squared brackets complies with quasiparticle expectations, namely mean potential times density, the second term displays the role of fluctuations in the potential energy density.

For specific interactions with the same number α of field operators attached to any vertex of $\hat{\mathcal{L}}^{\text{int}}$, one simply deduces

$$\mathcal{E}^{\text{int}}(X, p) = \frac{2}{\alpha} \mathcal{E}^{\text{pot}}(X, p). \quad (3.45)$$

Please also notice from (3.42) that the special combination

$$\Theta^{00}(X) + \frac{1}{3} \sum_{i=1}^3 \Theta^{ii}(X) = \text{Tr} \int \frac{d^4 p}{(2\pi)^4} A(X, p) f(X, p) \left[p_0 - \frac{2}{3} \epsilon_p^0 \right] \quad (3.46)$$

depends on the specific form of the interaction only via the spectral function. As we will see below, this combination simply relates to the entropy density in local thermodynamic equilibrium.

The conservation laws only hold, if all the self-energies are Φ -derivable. In ref. [8], it has been shown that this implies the following consistency relations obtained after Wigner transformation and first-order gradient expansion, (a) for the conserved current

$$i \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \left[\left\{ \text{Re}\Sigma^R, F \right\} - \left\{ \text{Re}G^R, \frac{\Gamma}{A} F \right\} + C \right] = 0, \quad (3.47)$$

and (b) for the energy-momentum tensor

$$\partial^\nu \left(\mathcal{E}^{\text{int}} - \mathcal{E}^{\text{pot}} \right) = \text{Tr} \int \frac{p^\nu d^4 p}{(2\pi)^4} \left[\left\{ \text{Re}\Sigma^R, F \right\} - \left\{ \text{Re}G^R, \frac{\Gamma}{A} F \right\} + C \right]. \quad (3.48)$$

The contributions from the Markovian collision term C drop out in both cases, cf. eq. (4.8) below. The first term in each of the two relations refers to the change from the free velocity \mathbf{v} to the group velocity \mathbf{v}_g , cf. eq. (3.32), in the medium. It can therefore be associated with a corresponding *drag-flow* contribution of the surrounding matter to the current or energy-momentum flow. The second (fluctuation) term compensates the former contribution and can therefore be associated with a *back-flow* contribution, which restores the Noether expressions (3.41) and (3.42) to be indeed the conserved quantities. In this compensation we see the essential role of the fluctuation term (3.30) in the generalized kinetic equation. Dropping or approximating this term would spoil the conservation laws. Indeed, both expressions (3.41) and (3.42) comply exactly with the generalized kinetic equation (3.27), i.e. they are exact integrals of the generalized kinetic equations.

Expressions (3.41) and (3.42) for 4-current and energy-momentum tensor, respectively, as well as self-consistency relations (3.47) and (3.48) are written explicitly for the case of nonrelativistic particles which number is conserved. This follows from the conventional way of nonrelativistic renormalization for such particles based on normal ordering. When the number of particles is not conserved (e.g., for phonons) or a system of relativistic particles is considered, one should replace $F(X, p) \rightarrow \frac{1}{2} (F(X, p) \mp \tilde{F}(X, p))$ in all above formulas in order to take proper account of zero point vibrations (e.g., of phonons) or of the vacuum polarization in the relativistic case. These symmetrized equations, derived from special (\mp) combinations of the transport equations (3.27) and (3.28), are generally ultra-violet divergent, and hence, have to be properly renormalized at the vacuum level.

3.5 Positive Definiteness of Kinetic Quantities

For a semi-classical interpretation one likes to have the Wigner distributions $F(X, p)$ and $\tilde{F}(X, p)$ to be positive semi-definite, hereto after just called "positive". Using the operator definition for the Green's functions (2.2) and integrating it over a large space-time volume, one arrives at, e.g.,

$$\int dX F(X, p) = \left\langle \left(\int dy e^{ipy} \hat{\varphi}^\dagger(y) \right) \left(\int dx e^{-ipx} \hat{\varphi}(x) \right) \right\rangle, \quad (3.49)$$

and similarly for self-energies Σ expressed through the current-current correlator. It indicates that the r.h.s. of such equality is real and nonnegative. Thus, we get the following set of constraints

$$\int dX \tilde{F}(X, p) \geq 0, \quad \int dX F(X, p) \geq 0, \quad \int dX \Gamma_{\text{out}}(X, p) \geq 0, \quad \int dX \Gamma_{\text{in}}(X, p) \geq 0. \quad (3.50)$$

Similar relations are obtained for the integration over four-momentum space rather than space and time. As a result, in stationary and spatially homogeneous systems, in particular in equilibrium systems, the quantities F , \tilde{F} , Γ_{in} and Γ_{out} are real and nonnegative, i.e.

$$F(p) \geq 0, \quad \tilde{F}(p) \geq 0; \quad \Gamma_{\text{in}}(p) \geq 0, \quad \Gamma_{\text{out}}(p) \geq 0. \quad (3.51)$$

In deriving constraints (3.50) and (3.51), we did not use the fact that the Green's functions are solutions of the Dyson's equation. However, we used the operator picture. Any approximation, in particular, if formulated in the space of Green's functions, may spoil such rigorous statements like (3.49). Nevertheless, both the Φ -derivable scheme and the gradient approximation preserve the retarded relations (A.6) among the different contour components and the retarded and advanced functions of any contour function, with definite values for the imaginary parts of the corresponding retarded Wigner functions

$$-2\text{Im}G^R(X, p) = A(X, p) \geq 0, \quad -2\text{Im}\Sigma^R(X, p) = \Gamma(X, p) \geq 0, \quad (3.52)$$

which even hold locally. In particular, solution (3.21) for the retarded Green's function shows that all retarded relations hold locally: the momentum part is the same as that in the homogeneous case with the space-time coordinate X as a parameter. Under the condition $|f - \gamma| \ll 1$, cf. the discussion around eq. (3.24), one finds that

$$\Gamma_{\text{out}}(X, p) \approx f\Gamma(X, p) > 0, \quad \Gamma_{\text{in}}(X, p) \approx (1 \mp f)\Gamma(X, p) > 0 \quad (3.53)$$

as long as the Wigner densities f and $1 \mp f$ are positive. As the gradient approximation is a quasi-homogeneous approximation, one may therefore expect the positivity of Γ_{in} and Γ_{out} to be preserved even in the self-consistent treatment discussed here. Diagrammatic

rules may also corroborate this, since diagrams for Γ_{in} and Γ_{out} are calculated like in the homogeneous case.

We now like to show that, if Γ_{in} and Γ_{out} are positive, also under minor restrictions the kinetic equations (3.27), (3.28) preserve the positivity of F and \tilde{F} , once one has started from positive F and \tilde{F} initially. For this purpose we turn to the generalized kinetic equation (3.27) for the phase-space distribution $F(X, p)$ which is of integro-quasi-linear first-order partial differential type in 8 dimensions. We present it in the form

$$\mathcal{D}F = A\Gamma_{\text{in}} - \Gamma F + \left\{ \Gamma \frac{F}{A}, \text{Re}G^R \right\}, \quad (3.54)$$

with the coefficient Z (cf. eq. (3.32)) in front of the time derivative being positive. The condition $Z > 0$ takes place due to the dispersion relation for the self-energy Σ^R and positive definiteness of $\Gamma(X, p)$, cf. eq. (3.52).

Given the solution, let us then discuss properties of F along characteristic curves determined by the quasi-linear drift operator \mathcal{D} with a curve parameter s growing monotonically with time. Assuming F is positive initially, the occurrence of a negative value at some later time requires the first zero value to occur at one of the characteristic curves. Be s_0 such a place, one finds

$$\frac{d}{ds} F(s) \Big|_{s=s_0} = \left[A\Gamma_{\text{in}} + \left\{ \Gamma \frac{F}{A}, \text{Re}G^R \right\} \right]_{s=s_0}. \quad (3.55)$$

If the r.h.s. is positive, one comes to a contradiction, as the approach from some positive value to a zero value which then becomes negative would require a nonpositive derivative at s_0 . Thus, $A\Gamma_{\text{in}} > |\left\{ \Gamma F/A, \text{Re}G^R \right\}|$ is a sufficient (not necessary) condition that F preserves its positivity. The corresponding condition $A\Gamma_{\text{out}} > |\left\{ \Gamma \tilde{F}/A, \text{Re}G^R \right\}|$ provides positivity of \tilde{F} . According to eq. (3.30) this puts restrictions on the size of the occurring gradients of the gain and loss rates along the characteristic curves, i.e. along the classical paths. This condition is met in a wide range of cases, namely (a) in the quasi-homogeneous and quasi-static case, i.e. close to equilibrium, where gradients are of order $C \ll A\Gamma_{\text{in}}, A\Gamma_{\text{out}}$, and (b) in the standard quasi-particle limit, where an extra small parameter, namely the width Γ , enters the estimations.

4 Collision Term

To further discuss the transport treatment we need an explicit form of the collision term (3.16), which is provided from the Φ functional in the $-+$ matrix notation via the variation rules (A.8) as

$$C(X, p) = \frac{\delta i\Phi}{\delta \tilde{F}(X, p)} \tilde{F}(X, p) - \frac{\delta i\Phi}{\delta F(X, p)} F(X, p). \quad (4.1)$$

Here we assumed Φ be transformed into the Wigner representation and all $\mp iG^{-+}$ and iG^{+-} to be replaced by the Wigner-densities F and \tilde{F} . Thus, the structure of the collision term can be inferred from the structure of the diagrams contributing to the functional Φ . To this end, in close analogy to the consideration of ref. [17], we discuss various decompositions of the Φ -functional, from which the in- and out-rates are derived. This consideration is based on the standard real-time diagrammatic rules, where the contour integrations are decomposed into two branches with $-$ and $+$ vertices for the time- and anti-time-ordered branches, cf. Appendix A. For the sake of physical transparency, we confine our treatment to the *local* case, where in Wigner representation all the Green's functions are taken at the same space-time coordinate X and all nonlocalities, i.e. derivative corrections, are disregarded. Derivative corrections give rise to memory effects in the collision term, which will be analyzed separately for the specific case of triangle diagram (see subsect. 4.4).

4.1 Diagrammatic Decomposition into Physical Sub-Processes

Consider a given closed diagram of Φ , at this level specified by a certain number n_λ of vertices and a certain contraction pattern which links all vertices with lines of certain arrow sense for complex fields. This fixes the topology of such a contour diagram. It leads to 2^{n_λ} different diagrams in the $-+$ notation from the summation over all $-+$ signs attached to each vertex. Thus for any $-+$ type diagram of Φ also the diagram, where all $+$ and $-$ vertex signs are interchanged, contributes to Φ . Furthermore, for any diagram with given line senses the diagram with all line senses reversed also is a valid diagram of Φ . We shall exploit these two discrete symmetry operations to further determine the properties of Φ . The simultaneous application of the interchange of all vertex signs and the reversion of the line sense leads to the adjoint expressions, since in the underlying operator picture it adjungates all operators and inverts the operator ordering. The corresponding values are then complex conjugate to one another, cf. (A.7).

Using these symmetries one can convert the functional Φ into the following general form

$$i\Phi = \frac{1}{2} \sum_{m,m'} (i\Phi_{m,m'} + i\Phi_{m',m}), \quad (4.2)$$

$$\begin{aligned} i\Phi_{m,m'} &= \int \frac{d^4 p_1}{(2\pi)^4} \dots \frac{d^4 p_m}{(2\pi)^4} \frac{d^4 p'_1}{(2\pi)^4} \dots \frac{d^4 p'_{m'}}{(2\pi)^4} \delta^4 \left(\sum_{i=1}^m p_i - \sum_{i=1}^{m'} p'_i \right) \\ &\times R_{m,m'}(X; p_1, \dots, p_m; p'_1, \dots, p'_{m'}) F_1 \dots F_m \tilde{F}'_1 \dots \tilde{F}'_{m'}, \end{aligned} \quad (4.3)$$

where in view of the local approximation the four momentum conservation has been extracted. While Φ_{00} compiles with terms void of any Wigner densities, i.e. from diagrams where all vertices have the same sign and which do not contribute to the collision term, the

nontrivial $\Phi_{m,m'}$ terms sum the sub-class of diagrams of Φ with precisely $m + m'$ Wigner densities $F_i = F(X, p_i)$ and $\tilde{F}'_i = \tilde{F}(X, p'_i)$, respectively. According to eq. (4.1) each $\Phi_{m,m'}$ in (4.3) generates a multi-particle gain or loss contributions expressed in terms of integrals over products of generalized distribution functions F , and Fermi/Bose factors \tilde{F} . Every term in the sum (4.2) has been duplicated, repeating each term in its line reverse form. The corresponding transition rates $R_{m,m'}(X; p_1, \dots, p_m; p'_1, \dots, p'_{m'})$ are real due to the adjunction symmetry. As explained below, they result from the product of chronological Feynman amplitudes, given by the sub-diagram compiled from all $-$ vertices linked by iG^{--} Green's functions times that of the anti-chronological part containing the iG^{++} functions.

The generic form (4.3) of $\Phi_{m,m'}$ can be illustrated diagrammatically. For this purpose consider a $-+$ notation diagram contributing to $\Phi_{m,m'}$, which contains vertices of either sign. It can be decomposed into two pieces, denoted by a compact bra-ket notation, say $(\alpha|$ and $|\beta)$, in such a way that each of the two sub-pieces contains vertices of only one type of sign³

$$\begin{aligned} i\Phi_{\alpha\beta}^{+-} &= \frac{1}{n_{\alpha\beta}} \left(\text{Diagram with } \alpha \text{ and } \beta \text{ vertices} \right) = \frac{1}{n_{\alpha\beta}} (\alpha | F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'} | \beta) \\ &= \frac{1}{n_{\alpha\beta}} \int \frac{d^{4(m+m')} p_{mm'}}{(2\pi)^{4(m+m')}} \delta^4 \left(\sum_{i=1}^m p_i - \sum_{i=1}^{m'} p'_i \right) \\ &\quad \times V_\alpha^*(X; p_{mm'}) F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'} V_\beta(X; p_{mm'}), \end{aligned} \quad (4.4)$$

where $n_{\alpha\beta}$ counts the number of vertices in the closed diagram $(\alpha|\beta)$. The short-hand notation $p_{mm'} = \{p_1, \dots, p_m; p'_1, \dots, p'_{m'}\}$ summarizes the momenta, type and internal quantum numbers of the set of ordered valences, to be joint to the product of $m + m'$ Wigner densities⁴ $F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'}$ linking the two amplitudes. The “end-caps” $(\alpha|$ and $|\beta)$ represent multi-point vertex functions, in simple cases of tree type, of only one sign for the vertices, i.e. they are either entirely time ordered ($-$ vertices) or entirely anti-time ordered ($+$ vertices). Each such vertex function

$$|\beta) = V_\beta(X; p_{mm'}), \quad (\alpha| = V_\alpha^*(X; p_{mm'}), \quad (4.5)$$

to be determined by normal Feynman diagram rules, has m F -valences and m' \tilde{F} -valences, respectively. In (4.5) we used the fact that adjoint expressions are complex conjugate to

³ To construct the decomposition, just deform a given mixed-vertex diagram of Φ in such a way that all $+$ and $-$ vertices are placed left and respectively right from a vertical division line and then cut along this line.

⁴ This product of Wigner-densities originates from the $\mp iG^{-+}$ and iG^{+-} Green's functions, (cf. eq. (3.7)). In closing the diagram by these Wigner densities extra fermion loops may appear besides the ones accounted for in the amplitudes α and β . Using the product $F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'}$ just take care of the additional (-1) factors arising from this closing of the diagram.

each other, cf. (A.7). Accumulating all diagrams of Φ that lead to the same set of Wigner densities $F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{\tilde{m}}$ provides us with the generic form (4.3) with the partial process rates

$$R_{m,m'}(X; p_{mm'}) = \sum_{(\alpha\beta) \in \Phi_{m,m'}} \frac{1}{n_{\alpha\beta}} \text{Re} \{ V_\alpha^*(X; p_{mm'}) V_\beta(X; p_{mm'}) \}. \quad (4.6)$$

The restriction to the real part arises, since with $(\alpha|\beta)$ also the adjoint $(\beta|\alpha)$ diagram contributes to the subclass $\Phi_{m,m'}$. However these rates are not necessarily positive as in perturbation theory⁵. In this point, the generalized scheme differs from the conventional Boltzmann kinetics. Even the two rates, $R_{mm'}$ and its line sense reverse $R_{m'm}$, have not to be necessarily identical to each other. Still eq. (4.3) represents the most general form of Φ expressed through the Wigner densities $F(X, p)$ and $\tilde{F}(X, p)$ in the local approximation. It is however important to realize that in many physically relevant cases, e.g. those discussed below, one indeed finds that

$$R_{m,m'} = R_{m',m}. \quad (4.7)$$

This property will be used as a sufficient condition for the derivation of the H-theorem. In the following we will restrict the discussion to cases where (4.7) is assumed. The treatment of more general cases, in particular in connection with the H-theorem, will be deferred to another publication.

Since Φ is two-line irreducible, there are at least three lines connecting $(\alpha|$ and $|\beta)$ and in many cases $(\alpha|$ and $|\beta)$ are connected by, at least, four lines, like in the case of two-body interactions. In such diagrams each of the amplitudes α or β necessarily form a connected diagram for the complex field case for binary ($m = m' = 2$) and for triple scattering ($m = m' = 3$) case, while in the general case of multi-particle (more than triple) processes such amplitudes may be disconnected.

No further symmetry can be specified at this level without additional knowledge on possible topological and other symmetries of Φ and the different particle species involved. The decomposition discussed here solely relies on a straightforward application of the contour rules for multi-point functions. They differ from other cutting rules like those derived by Danielewicz [29], which represent the result in terms of generalized retarded functions.

⁵ In perturbation theory they are positive, since there the sum in (4.6) leads to absolute squares. In the general case with resummed propagators this positivity may be lost due to the restriction to diagrams of Φ which are globally two-particle irreducible. The latter excludes certain combinations of amplitude diagrams, implying that the rates of genuine multi-particle processes are not necessarily positive.

4.2 Local Collision Term and Memory Corrections

The gradient corrections to the folding of the self-energies with the propagators in the collision term have already been accounted for in the kinetic equations. They essentially give rise to in-medium corrections of the convective part (l.h.s) of the generalized kinetic equation (3.27). Within the spirit of the gradient approximation one also likes to express the self-energies themselves by space-time local quantities. Thus, for a consistent gradient approximation further gradient corrections are admissible once the self-energy diagrams and thereby the diagrams of Φ consist of more than 3 vertices. We call the collision term, evaluated with all Green's functions in the Wigner representation taken at the same space-time point X , the *local* collision term.

In terms of the representation (4.3) of Φ and implying line-sense reversal symmetry for the rates (4.7) the matrix variation rules (4.1) determine the following local part of the collision term (3.16) for a particle of flavor "a" as

$$\begin{aligned} C_a^{\text{loc}}(X, p) &= \frac{1}{2} \sum_{m,m'} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} \frac{d^4 p'_1}{(2\pi)^4} \cdots \frac{d^4 p'_{m'}}{(2\pi)^4} R_{m,m'}(X; p_1, \dots, p_m; p'_1, \dots, p'_{m'}) \\ &\times \left\{ \tilde{F}_1 \cdots \tilde{F}_m F'_1 \cdots F'_{m'} - F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'} \right\} \\ &\times \left[\sum_{i=1}^m \delta_{aa_i} \delta^4(p_i - p) - \sum_{i=1}^{m'} \delta_{aa_i} \delta^4(p'_i - p) \right] \delta^4 \left(\sum_{i=1}^m p_i - \sum_{i=1}^{m'} p'_i \right). \end{aligned} \quad (4.8)$$

The functional variations of Φ with respect to F_a and \tilde{F}_a are expressed in terms of the four-momentum projectors $\delta_{aa_i} \delta^4(p_i - p)$ which in case of multi-component systems of particles with different flavors and internal quantum numbers "a" also include the proper projections onto the different flavors $a_i = a$. This expression nicely visualizes the detailed balance property namely that the same multi-particle rate determines both the forward and the backward processes.

The first-order gradient corrections to the local collision term (4.8) we call *memory* corrections. We refrain from specifying the latter in general terms. They are linear in the spatial gradients of Wigner functions and the displacement factors in front of these Taylor expansions give momentum gradients of associated other multi-point functions. An explicit example will be discussed in section 4.4. Often (but not in general) there are special reasons why C_{mem} is small. In particular, *only diagrams of third and higher order in the number of vertices give rise to memory effects*, which are discussed below (sect. 4.4) within a simple model. One has to keep in mind that memory effects are quite commonly neglected in transport models.

4.3 Φ -Derivable Collision Terms

4.3.1 Two-body Potential Interaction

To be specific we consider a system of fermions interacting via a two-body potential $V = V_0\delta(x - y)$, and, for the sake of simplicity, disregard its spin structure, by relating spin and anti-symmetrization effects to a degeneracy factor d . To derive the decomposition of a Φ -derivable collision term, we employ the rules as described in subsect. 4.4.

In the first example, we consider the generating functional Φ to be approximated by the following two diagrams

$$i\Phi = \frac{1}{2} \begin{array}{c} \text{Diagram of two circles connected by a dashed line} \end{array} + \frac{1}{4} \begin{array}{c} \text{Diagram of a double loop with a dashed line through the center} \end{array} \quad (4.9)$$

with the dashed line illustrating the decomposition according to (4.4). Here the $1/n_\lambda$ factors start with $1/2, 1/4, \dots$ according to the nonrelativistic diagram rules for two-body interaction, cf. Appendix C, i.e. the vertex dots are considered as the zero range limit of a finite range interaction. In the $\{-+\}$ matrix notation of the Green's functions, one can easily see that the one-point diagram does not contribute to the collision term, while decomposing the second diagram along the dashed line leads to a purely local result

$$\begin{aligned} C^{(2)} &= d^2 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \left| \begin{array}{c} \text{X} \\ \text{---} \end{array} \right|^2 \\ &\times \delta^4(p + p_1 - p_2 - p_3) \left(F_2 F_3 \tilde{F} \tilde{F}_1 - \tilde{F}_2 \tilde{F}_3 F F_1 \right), \end{aligned} \quad (4.10)$$

where the brief notation of the previous subsect. is used for F_i and \tilde{F}_i . This collision integral has precisely the form of the binary collision term of Boltzmann–Uehling–Uhlenbeck (BUU), except for the fact that distribution functions are not constrained to the mass shell. The binary transition rate

$$R_2^{(2)} = V_0^2 = \left| \begin{array}{c} \text{X} \\ \text{---} \end{array} \right|^2 \quad (4.11)$$

is nonnegative in this case. Here and below, the bracketed superscript ⁽²⁾ (or ⁽³⁾) points out the origin of the quantity (C , R , etc.) from the second (or third) diagram of Φ . The subscript ₂ in the transition rate of eq. (4.11) indicates the binary-collision nature of this transition rate. Note that external 4-momenta (in-going and out-going) of the scattering amplitude X are not confined to the mass shell. For the trivial case under consideration, this fact does not give rise to any important consequences. However, for more complicated examples below, it means that the collision term is determined by off-shell scattering amplitudes.

The picture becomes more complicated, if Φ involves diagrams of higher orders. For instance, let us add the following three point diagram to Φ , i.e.

$$\begin{aligned} i\Phi &= i(\Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)}) \\ &= \frac{1}{2} \text{Diagram 1} + \frac{1}{4} \text{Diagram 2} + \frac{1}{6} \text{Diagram 3}, \end{aligned} \quad (4.12)$$

where one possible decomposition is illustrated by dashed lines. The corresponding self-energy becomes

$$\begin{aligned} -i\Sigma &= -i(\Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}) = \\ &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}. \end{aligned} \quad (4.13)$$

Now the collision term contains a *nonlocal* part due to the last diagram. This nonlocal contribution is discussed in the next subsect. in detail. The local part can easily be derived in the form

$$\begin{aligned} C^{(2)} + C_{\text{loc}}^{(3)} &= d^2 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \left(\left| \text{Diagram 1} + \text{Diagram 2} \right|^2 - \left| \text{Diagram 3} \right|^2 \right) \\ &\times \delta^4(p + p_1 - p_2 - p_3) (F_2 F_3 \tilde{F} \tilde{F}_1 - \tilde{F}_2 \tilde{F}_3 F F_1), \end{aligned} \quad (4.14)$$

where all the vertices in the off-shell scattering amplitudes are of the same sign, say “–” for definiteness, i.e. there are no “+–” and “–+” Green’s functions left. The quantity $C^{(2)} + C_{\text{loc}}^{(3)}$ is again of the Boltzmann form with the transition rate

$$R_2^{(2)} + R_2^{(3)} = \left| \text{Diagram 1} + \text{Diagram 2} \right|^2 - \left| \text{Diagram 3} \right|^2. \quad (4.15)$$

At the first glance, one may argue that this rate is not necessarily positive in the limit of strong coupling. Indeed, the first term in eq. (4.15), i.e. $-iV_0$, is purely imaginary, whereas the second one—the loop—given by

$$\text{loop} = -d \int \frac{d^4 p_1}{(2\pi)^4} |V_0|^2 iG^{--}(x, p + p_1) iG^{--}(x, p_1),$$

has both real and imaginary parts. Hence, the real part of this loop is canceled out in eq. (4.15). If $| \text{Im}(\text{loop}) | > | V_0 |$, the rate $(R_2^{(2)} + R_2^{(3)})$ may become negative, depending on signs of V_0 and $\text{Im}(\text{loop})$. However, one has to keep in mind that the Green's functions in the loop cannot be chosen arbitrarily. Rather in a consistent treatment, as shown in eq. (80) of ref. [17], the loop reveals a factor $\propto |1/\Gamma| \propto |1/V_0^2|$ resulting from the imaginary part of the retarded self-energy in the propagators, which balances the total value of the loop. Indeed, in equilibrium the gain part of collision term and thus $(R_2^{(2)} + R_2^{(3)})$ is positive. This illustrates that the question of positive definiteness is quite subtle.

4.3.2 Bosonization of the Interaction

It is obvious that the situation becomes complicated once one extends the picture to ring diagrams with more than three vertices. Yet, there is a simple (however, not general) strategy to proceed. We may avoid the two-particle potential interaction from the very beginning and rather introduce an interaction mediated by an artificially introduced neutral heavy scalar boson (h.b.) of mass $m_{\text{h.b.}}$ much larger than any characteristic momentum transfers in the system. Then the free retarded Green's function of the boson approximately equals

$$\Delta_{\text{h.b.}}^{0R} \simeq \frac{-1}{m_{\text{h.b.}}^2 - i0}, \quad (4.16)$$

and the vertex of fermion–heavy-boson interaction being $g = \sqrt{|V_0|m_{\text{h.b.}}}$. Moreover, by the same reason the heavy-boson occupation numbers may be put to zero, $\Delta_{\text{h.b.}}^{0-+} = 0$, since typical excitation energies are much less than the boson mass. The fact that this boson is very heavy makes the fermion–fermion interaction almost point-like. To be specific, we assume that $V_0 < 0$, i.e. attractive, which can be mediated by a scalar boson. In case of repulsive interactions, a vector boson would be an appropriate choice.

Thus, *from now on, we deal with a system of interacting fermions and heavy bosons*. Let us take the following approximation for the corresponding Φ functional (we call it $\Phi_{\text{h.b.}}$)

$$i\Phi_{\text{h.b.}} = \frac{1}{2} \cdot \text{Diagram} \quad (4.17)$$

in terms of full Green's functions of fermions (the bold solid lines) and bosons (the bold

wavy line). In this approximation, the boson self-energy is given by (cf. eq. (2.9))⁶

$$-\mathrm{i}\frac{1}{2}\Sigma_{\text{h.b.}} = \frac{\delta\mathrm{i}\Phi_{\text{h.b.}}}{\delta\mathrm{i}G_{\text{h.b.}}} = \text{Diagram with a wavy line and a loop} \quad (4.18)$$

and the heavy-boson Green's function is defined by the standard Dyson's equation

$$\mathrm{i}G_{\text{h.b.}} = \text{Diagram with a wavy line} = \text{Diagram with a wavy line} + \text{Diagram with a wavy line and a loop}. \quad (4.19)$$

We would like to compare this model with the model described in subsection 4.3.1. Eliminating the artificial heavy boson one effectively sums up all ring diagrams of the type

$$\mathrm{i}\Phi_{\text{ring}} = \frac{1}{2} \text{Diagram with two loops} + \frac{1}{4} \text{Diagram with three loops} + \frac{1}{6} \text{Diagram with four loops} + \frac{1}{8} \text{Diagram with five loops} + \dots \quad (4.20)$$

Note that $\Phi_{\text{ring}} \neq \Phi_{\text{h.b.}}$, since the summations of loops in eq. (4.17) and eq. (4.20) have different sense. In eq. (4.19) we have the conventional diagrammatic summation, while in eq. (4.20) the summation is logarithmic, i.e. with the factors $1/n$, where n is a number of vertices in a diagram. However, this difference in summations is compensated for by the heavy-boson contribution to the generating functional (2.8). The equivalence of approximations (4.20) and (4.17) can be actually seen, e.g., from the fact that they result in the same approximation for the fermion self-energy $\mathrm{i}\Sigma_{\text{f}} = \delta\Phi_{\text{ring}}/\delta G_{\text{f}} = \delta\Phi_{\text{h.b.}}/\delta G_{\text{f}}$ (cf. eq. (2.9)) after substituting eq. (4.19) for the heavy-boson propagator. Here and below, the sub-label "f" denotes fermion quantities.

Upon introduction of the heavy-boson degree of freedom, we have to deal with two coupled transport equations—for nonrelativistic fermions and for heavy bosons—with the following collision terms

$$\begin{aligned} C_{\text{f}}(X, p) &= \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \frac{\mathrm{d}^4 p_2}{(2\pi)^4} g^2 \delta^4(p - p_1 - p_2) \\ &\times \left[\tilde{F}_{\text{f}}(X, p) F_{\text{f}}(X, p_1) F_{\text{h.b.}}(x, p_2) - F_{\text{f}}(X, p) \tilde{F}_{\text{f}}(X, p_1) \tilde{F}_{\text{h.b.}}(x, p_2) \right], \end{aligned} \quad (4.21)$$

for fermions and

⁶ Note that for neutral bosons, which number is not conserved, the additional factor 1/2 appears in eq. (2.9), cf. ref. [8].

$$C_{\text{h.b.}}(X.p) = d \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} g^2 \delta^4(p + p_1 - p_2) \\ \times \left[\tilde{F}_{\text{h.b.}}(X, p) \tilde{F}_{\text{f}}(X, p_1) F_{\text{f}}(X, p_2) - F_{\text{h.b.}}(X, p) F_{\text{f}}(X, p_1) \tilde{F}_{\text{f}}(X, p_2) \right], \quad (4.22)$$

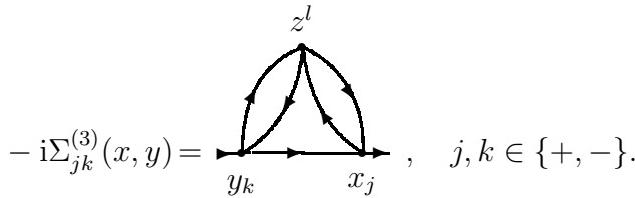
for heavy bosons, where $g^2 = -V_0 m_{\text{h.b.}}^2 > 0$ is defined in terms of two-particle interaction strength $V_0 < 0$ and the heavy boson mass $m_{\text{h.b.}}$.

The collision terms (4.21) and (4.22) are very simple in spite of the fact that they involve the whole series of ring diagrams. The corresponding gain and loss terms are positive and contain no memory effects, as they are hidden in the boson, while in the pure fermionic case already the triangle diagram gives rise to memory effects. Indeed, there is no contradiction here. If one wants to eliminate the bosonic degree of freedom, one has to resolve the bosonic transport equation with respect to the bosonic generalized distribution function $F_{\text{h.b.}}$ for entire past and substitute this into the fermionic collision term. In this way, the resulting collision term becomes highly complicated and nonlocal and thus contains memory effects.

Hence, we have demonstrated that sometimes it is useful to introduce new degrees of freedom in order to achieve a reasonable collision term. Of course, all the consideration above remains valid also for particle-particle interaction mediated by a real boson rather than only by artificially introduced one.

4.4 Memory Effects in Collision Term

A general treatment of memory effects in the collision term is a cumbersome task. In this subsection we continue to consider a system of nonrelativistic fermions interacting via the contact two-body potential and concentrate on the third diagram in eq. (4.12), which has already been considered in a Φ -derivable scheme for the thermodynamic potential and entropy in refs [14,15]. This is the first ring diagram to contribute to memory effects. The corresponding self-energy diagram reads



$$-i\Sigma_{jk}^{(3)}(x, y) = \text{Diagram} \quad , \quad j, k \in \{+, -\}. \quad (4.23)$$

Standard diagrammatic rules in the matrix representation present $\Sigma_{ij}^{(3)}$ in analytic form as

$$-i\Sigma_{jk}^{(3)}(x, y) = \int dz \cdot iV_0 \sigma_{jj'} \sigma_{kk'} iG^{j'k'}(x, y) L^{k'l}(y, z) \sigma_{ll'} L^{l'j'}(z, x), \quad (4.24)$$

where we have introduced a loop function

$$L^{jk}(x, y) = \text{diag} \circ \text{G}^{jk}(x, y) \text{G}^{kj}(y, x), \quad (4.25)$$

and σ_{ij} are given by eq. (A.2). As above, the factor d results from the trace over spin. In the Wigner representation, L^{jk} takes the form

$$L^{jk}(X, p') = \int \frac{d^4 p''}{(2\pi)^4} \tilde{L}^{jk}(X; p'' + p', p''), \quad (4.26)$$

where

$$\tilde{L}^{jk}(X; p'' + p', p'') = dV_0 iG^{jk}(X, p'' + p') iG^{kj}(X, p''). \quad (4.27)$$

The loop functions L possess notable properties

$$L^{++} + L^{--} = L^{-+} + L^{+-}, \quad L^{jk}(x, y) = L^{kj}(y, x), \quad L^{jk}(X, p) = L^{kj}(X, -p). \quad (4.28)$$

The former property follows from the general property (A.6) of the two-point functions and holds in both the coordinate and Wigner representations. Proceeding from eq. (4.24) and with the help of relation (3.5), we can immediately evaluate the Wigner transform of $\Sigma_{ij}^{(3)}$ (cf. eq. (3.1)) and perform its gradient expansion

$$\Sigma_{jk}^{(3)} \simeq (\Sigma_{jk}^{(3)})_{\text{loc}} + (\Sigma_{jk}^{(3)})_{\text{mem}}, \quad (4.29)$$

where

$$-i(\Sigma_{jk}^{(3)})_{\text{loc}}(X, p) = \int \frac{d^4 p'}{(2\pi)^4} iV_0 \sigma_{jj'} \sigma_{kk'} \sigma_{ll'} iG^{j'k'}(X, p' + p) L^{k'l}(X, p') L^{l'j'}(p', X) \quad (4.30)$$

is the local contribution to $\Sigma_{ij}^{(3)}$, and

$$\begin{aligned} -i(\Sigma_{jk}^{(3)})_{\text{mem}}(X, p) &= \frac{i}{2} \int \frac{d^4 p'}{(2\pi)^4} iV_0 \sigma_{jj'} \sigma_{kk'} \sigma_{ll'} iG^{j'k'}(X, p' + p) \\ &\times \left\{ L^{k'l}(X, p'), L^{l'j'}(X, p') \right\}_{p', X} \end{aligned} \quad (4.31)$$

is the first-order gradient (memory) correction, which is of central interest in this subsection. Here, the Poisson bracket is taken with respect to the (p', X) variables indicated in the subscript.

To evaluate the memory correction for the collision term (cf. eqs (3.9) and (3.16)), we need to consider the following combinations of self-energies with Green's functions

$$\left(\Sigma_{+-}^{(3)}\right)_{\text{mem}}(X, p) G^{-+}(X, p) = \frac{i}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{d} \tilde{L}^{+-}(X; p' + p, p) \sigma_{ll'} \left\{ L^{-l}, L^{l'} \right\}_{p', X}, \quad (4.32)$$

$$G^{+-}(x, p) \left(\Sigma_{-+}^{(3)}\right)_{\text{mem}}(X, p) = \frac{i}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{d} \tilde{L}^{-+}(X; p' + p, p) \sigma_{ll'} \left\{ L^{+l}, L^{l'-} \right\}_{p', X}. \quad (4.33)$$

With the symmetry relations

$$\sigma_{ll'} \left\{ L^{-l}, L^{l'} \right\}_{p', X} = \sigma_{ll'} \left\{ L^{+l}, L^{l'-} \right\}_{p', X} = \left\{ L^{+-}, L^{-+} \right\}_{p', X} \quad (4.34)$$

deduced from (4.28) one determines the first-order gradient correction to the collision term induced by graph (4.23) as

$$\begin{aligned} C_{\text{mem}}^{(3)}(X, p) &= \left[\left(\Sigma_{+-}^{(3)}\right)_{\text{mem}}(X, p) G^{-+}(X, p) - G^{+-}(X, p) \left(\Sigma_{-+}^{(3)}\right)_{\text{mem}}(X, p) \right] \\ &= \frac{i}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{d} \left[\tilde{L}^{+-}(X; p' + p, p) - \tilde{L}^{-+}(X; p' + p, p) \right] \left\{ L^{+-}, L^{-+} \right\}_{p', X}. \end{aligned} \quad (4.35)$$

The corresponding local collision term is given by eq. (4.14).

5 H-Theorem

5.1 Time-Irreversibility of the Generalized Kinetic Description

Compared to exact descriptions, which are time reversible, reduced description schemes in terms of relevant degrees of freedom have access only to some limited information and thus normally lead to irreversibility. Various reduction and coarse graining schemes have been discussed and developed over the years. In the Green's function formalism presented here the information loss arises from the truncation of the exact Martin–Schwinger hierarchy, where the exact one-particle Green's function couples to the two-particle Green's functions, cf. [6,22], which in turn are coupled to the three-particle level, etc. ⁷ This truncation is achieved by the standard Wick decomposition, where all observables are expressed through one-particle propagators and therefore higher-order correlations are dropped. This step provides the Dyson equation (2.3) and the corresponding loss of information is expected to lead to a growth of entropy with time.

At the operator level the nonequilibrium entropy can be straightforwardly formulated in terms of von Neumann's entropy, which is an entropy in the information theory sense [30]

⁷ The appropriate classical limit leads to the Bogolyubov–Born–Green–Kirkwood–Yvon chain of equations.

$$S = -\text{Tr} \hat{\rho} \ln \hat{\rho}. \quad (5.1)$$

It is given by the expectation value of the logarithm of the density operator $\hat{\rho}$ itself. The problem of the Green's function formalism is that it does not give a direct access to the density operator itself but rather describes the space-time dependence of the expectation values of well defined operators, say $\langle \hat{A}(t) \rangle = \text{Tr} \hat{A} \hat{\rho}(t)$. To this extend, there is no immediate formulation of the entropy (5.1) in terms of Green's functions for nonequilibrium case. The situation is different at equilibrium, where with $\hat{\rho} = \exp(-\beta(\hat{H} - \mu \hat{N}))/Z$ a well defined density operator exists, which leads to the well known Matsubara or real-time formulation of the equilibrium entropy, cf. sect. 6 below. As will be seen, even in equilibrium, the entropy expression is not priori given but rather depends on the choice of Φ in a Φ -derivable scheme.

In order to access a nonequilibrium expression relevant for our generalized transport equation (3.27), we shall start from this transport equation and derive a flow expression s^μ with the property that its divergence grows in time, i.e.

$$\partial_\mu s^\mu(X) \geq 0, \quad (5.2)$$

and which in the equilibrium limit merges the corresponding equilibrium form of the entropy flow. That is, we show the existence of an H -theorem for our generalized kinetic description and $s^\mu(X)$ is thus identified with the kinetic entropy flow. Thereby, the validity conditions for the derivation of this kinetic entropy flow coincide with those of the kinetic equations themselves.

5.2 Markovian Entropy Flow

We start with general manipulations which lead us to definition of the kinetic entropy flow. We multiply eq. (3.28) by $(\mp) \ln(\tilde{F}/A)$, eq. (3.27) by $-\ln(F/A)$, take their sum, integrate it over $d^4 p/(2\pi)^4$ and finally sum the result over internal degrees of freedom like spin (Tr). Using the identity for the Poisson brackets

$$\{B, Af\} \ln f \pm \{B, A(1 \mp f)\} \ln(1 \mp f) = \{B, Af \ln f \pm A(1 \mp f) \ln(1 \mp f)\} \quad (5.3)$$

for any functions A, B and positive f and $1 \mp f$, one then arrives at the following relation

$$\partial_\mu s_{\text{loc}}^\mu(X) = \text{Tr} \sum_a \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}_a}{F_a} C_a(X, p), \quad (5.4)$$

where the quantity

$$s_{\text{loc}}^\mu = \sum_a s_{\text{loc},a}^\mu = \text{Tr} \sum_a \int \frac{d^4 p}{(2\pi)^4} \left[\left(v^\mu - \frac{\partial \text{Re}\Sigma_a^R}{\partial p_\mu} \right) \left(\mp \tilde{F}_a \ln \frac{\tilde{F}_a}{A_a} - F_a \ln \frac{F_a}{A_a} \right) - \text{Re}G_a^R \left(\mp \ln \frac{\tilde{F}_a}{A_a} \frac{\partial}{\partial p_\mu} \left(\Gamma_a \frac{\tilde{F}_a}{A_a} \right) - \ln \frac{F_a}{A_a} \frac{\partial}{\partial p_\mu} \left(\Gamma_a \frac{F_a}{A_a} \right) \right) \right] \quad (5.5)$$

obtained from the l.h.s. of the kinetic equation is interpreted as the local (Markovian) part of the entropy flow. Here we have restored the summation over "a" denoting the different particle species and intrinsic quantum numbers for a multi-component system. It illustrates that, although this entropy expression accounts for interactions among all particles, it can be expressed as a sum of the individual contributions, each of which is solely determined by the particle self-energy $\text{Re}\Sigma_a^R$ and its width Γ_a . Gradient corrections to the collision term (i.e. C_{mem}) give rise to extra memory contributions to the entropy flow.

Partial integrations in eq. (5.5) lead us to a more transparent expression for the entropy flow

$$s_{\text{loc}}^\mu = \text{Tr} \sum_a \int \frac{d^4 p}{(2\pi)^4} A_{sa}^\mu(X, p) \sigma_a(X, p), \quad \sigma_a(X, p) = \mp [1 \mp f] \ln [1 \mp f] - f \ln f, \quad (5.6)$$

where

$$A_{sa}^\mu(X, p) = \frac{A_a \Gamma_a}{2} B_a^\mu \quad (5.7)$$

has the meaning of an entropy-flow spectral function, while

$$B_a^\mu = A_a \left[\left(v^\mu - \frac{\partial \text{Re}\Sigma_a^R}{\partial p_\mu} \right) - M_a \Gamma_a^{-1} \frac{\partial \Gamma_a}{\partial p_\mu} \right], \quad (5.8)$$

is the flow spectral function, cf. the corresponding drift term (proportional to $\partial_\mu f$ in eq. (3.24)). The zero-components of these functions coincide with the corresponding two functions introduced in refs [31,32] for the case of equilibrium systems. Moreover, the B^0 function satisfies the sum rule

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} B_a^0 = 1, \quad (5.9)$$

which is easily obtained from the sum rule (3.23) for the spectral function A with the help of Kramers–Kronig dispersion relations. For the case that the considered particle a is a resonance, like the Δ or ρ -meson resonances in hadron physics, the B^0 function relates to the energy variations of scattering phase shift of the scattering channel coupling to the resonance in the virial limit, for details see e.g. refs [31,32,8].

In the noninteracting-particle limit, the entropy (the zero component of s_{loc}^μ) directly transforms into the proper ideal gas expression, cf. ref. [33]. In the quasiparticle approximation, the Γ_{in} and Γ_{out} terms have an additional smallness, which allows to neglect these terms. Thus, expression (5.5) for the entropy flow takes the form

$$(s_{\text{loc}}^\mu)^{\text{qp}} = \begin{pmatrix} s_0 \\ \mathbf{s} \end{pmatrix} = \text{Tr} \sum_a \int \frac{d^3 p}{(2\pi)^3} \begin{pmatrix} 1 \\ \frac{\partial \varepsilon_a}{\partial \mathbf{p}} \end{pmatrix} \left[\mp (1 \mp f_a^{\text{qp}}) \ln (1 \mp f_a^{\text{qp}}) - f_a^{\text{qp}} \ln f_a^{\text{qp}} \right] \quad (5.10)$$

in the quasiparticle limit, which follows from the substitution of eqs (3.35) and (3.36) into eq. (5.5). From (5.6) eq. (5.10) is also easily recovered, since $A^2 \Gamma / 2$ transforms to the corresponding δ -function in the limit $\Gamma \rightarrow 0$.

To prove explicitly the H -theorem we have to show that the r.h.s. of eq. (5.4) is nonnegative. To this end, we should consider the convolution of the collision term with $\ln(\tilde{F}/F)$. First, we do this for collision terms in local approximation.

5.3 Local Collision Term and H -Theorem

Using the multi-particle process decomposition (4.8) we arrive at the relation

$$\begin{aligned} \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{loc}}(X, p) &= \text{Tr} \sum_{m, m'} \frac{1}{2} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} \frac{d^4 p'_1}{(2\pi)^4} \cdots \frac{d^4 p'_{m'}}{(2\pi)^4} \\ &\times \left\{ F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'} - \tilde{F}_1 \cdots \tilde{F}_m F'_1 \cdots F'_{m'} \right\} \ln \frac{F_1 \cdots F_m \tilde{F}'_1 \cdots \tilde{F}'_{m'}}{\tilde{F}_1 \cdots \tilde{F}_m F'_1 \cdots F'_{m'}} \\ &\times R_{m, m'} \delta^4 \left(\sum_{i=1}^m p_i - \sum_{i=1}^{m'} p'_i \right). \end{aligned} \quad (5.11)$$

Here we assumed different flavors and intrinsic quantum number to be absorbed in the momenta p_i and p'_i . In the case when all rates $R_{m, m'}$ are nonnegative, i.e. $R_{m, m'} \geq 0$, this expression is nonnegative, since $(x - y) \ln(x/y) \geq 0$ for any positive x and y . In particular, $R_{m, m'} \geq 0$ takes place for all Φ -functionals up to two vertices. Then the divergence of s_{loc}^μ is nonnegative which proves the H -theorem in this case with (5.5) as the nonequilibrium entropy flow.

5.4 Explicit examples for the H -Theorem

We explicitly discuss the two examples introduced already in sect. 4.3 with Markovian collision terms, i.e. with the Φ functional consisting of one- and two-point diagrams only.

In the pure fermionic case with collision term (4.9), one can state an exact H -theorem

$$\begin{aligned} \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C^{(2)} &= d^3 \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} |\not{x}|^2 \\ &\times \delta^4(p + p_1 - p_2 - p_3) \ln \frac{\tilde{F}\tilde{F}_1F_2F_3}{FF_1\tilde{F}_2\tilde{F}_3} (\tilde{F}\tilde{F}_1F_2F_3 - FF_1\tilde{F}_2\tilde{F}_3) \geq 0. \end{aligned} \quad (5.12)$$

Note that here and below the extra factor d comes from summation (Tr) over spin of fermions.

Furthermore, it is instructive to consider the Φ approximation (4.17), where a heavy scalar boson has been introduced in order to sum up the series of ring diagrams of eq. (4.20), cf. subsect. 4.3.2. In this case, there are two coupled kinetic equations with collision terms (4.21) for fermions and (4.22) for heavy bosons, leading to

$$\partial_\mu s_{\text{loc}}^\mu = \partial_\mu s_f^\mu + \partial_\mu s_{\text{h.b.}}^\mu = \int \frac{d^4 p}{(2\pi)^4} \left(d \ln \frac{\tilde{F}_f}{F_f} C_f + \frac{1}{2} \ln \frac{\tilde{F}_{\text{h.b.}}}{F_{\text{h.b.}}} C_{\text{h.b.}} \right), \quad (5.13)$$

where s_{loc}^μ is given by the sum of the proper fermion (s_f^μ) and heavy-boson ($s_{\text{h.b.}}^\mu$) contributions. The thermodynamic entropy for this system, i.e. for Φ given by diagram (4.17), has recently been obtained in ref. [39]. Here we present the corresponding nonequilibrium entropy flow together with an affirmation of the H -theorem also for this case. The collision term, indeed, becomes

$$\begin{aligned} \partial_\mu s_{\text{loc}}^\mu &= d \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_b}{(2\pi)^4} g^2 \delta^4(p_1 - p_2 - p_b) \\ &\times \ln \frac{\tilde{F}_1 F_2 F_{\text{h.b.}}}{F_{f1} \tilde{F}_{f2} \tilde{F}_{\text{h.b.}}} (\tilde{F}_{f1} F_{f2} F_{\text{h.b.}} - F_{f1} \tilde{F}_{f2} \tilde{F}_{\text{h.b.}}) \geq 0, \end{aligned} \quad (5.14)$$

which is nonnegative.

Our representation of the entropy of a system interacting via two-body potential (i.e. as a sum of a purely fermionic part and that of the artificially introduced heavy boson) is also very similar to that derived by Riedel [14] within the ring-diagram model of Φ -derivable thermodynamics. In both cases, the bosonic part of the entropy $s_{\text{h.b.}}^\mu$ takes account of the fermion–fermion interaction calculated within the ring-diagram approximation (4.20). In thermodynamics, this interaction part of the entropy gives rise to the famous correction to the specific heat of liquid ${}^3\text{He}$ [14–16]: $\sim T^3 \ln T$, where T is the temperature. As has been found by Carneiro and Pethick [15], this correction to the specific heat emerges already solely from the third diagram of the whole ring series (4.20). To demonstrate the same within our kinetic approach, we should consider the Φ -derivable model (4.12) involving only the first three ring diagrams. Moreover, since the local entropy expression

(cf. eq. (5.5)) derived above does not contain such kind of corrections, one has to explicitly consider memory effects in the collision term (4.35).

5.5 Memory Effects in Entropy Flow and H-Theorem

We assume that the fermion–fermion potential interaction is such that the corresponding transition rate (4.15) is always nonnegative, so that the H -theorem takes place in the local approximation, i.e. when we keep only $C^{(2)} + C_{\text{loc}}^{(3)}$. Our aim now is to derive the entropy, which takes into account memory effects in the collision term ($C_{\text{mem}}^{(3)}$).

Proceeding similarly to that in subsect. 5.2, we multiply eq. (3.28) by $-\ln(\tilde{F}/A)$, eq. (3.27) by $-\ln(F/A)$, make their sum and integrate it over $d^4p/(2\pi)^4$. Thus, we arrive at the equation

$$\partial_\mu s_{\text{loc}}^\mu(X) = \text{Tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} (C^{(2)} + C_{\text{loc}}^{(3)}) + \text{Tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{mem}}^{(3)}, \quad (5.15)$$

where s_{loc}^μ is still the Markovian entropy flow defined by eq. (5.5). Our aim here is to present the last term on the r.h.s. of eq. (5.15) in the form of full x -derivative

$$\text{Tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{mem}}^{(3)} = -\partial_\mu s_{\text{mem}}^\mu(X) + \delta c_{\text{mem}}(X) \quad (5.16)$$

of some function $s_{\text{mem}}^\mu(X)$, which we then interpret as a non-Markovian correction to the entropy flow of eq. (5.5), plus a correction (δc_{mem}) which is small in some sense. Indeed, this term on the r.h.s. of eq. (5.16) is linear in X - and p -derivatives. Hence, it cannot be transformed into sign-definite form. The only possibility remained is to construct a full derivative of it. If we succeed to find a proper $s_{\text{mem}}^\mu(X)$, then relying on smallness of δc_{mem} we obtain

$$\partial_\mu (s_{\text{loc}}^\mu + s_{\text{mem}}^\mu) \simeq \text{Tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} (C^{(2)} + C_{\text{loc}}^{(3)}) \geq 0, \quad (5.17)$$

which is the H -theorem for the non-Markovian kinetic equation under consideration with $s_{\text{loc}}^\mu + s_{\text{mem}}^\mu$ as the proper entropy flow. The r.h.s. of eq. (5.17) is nonnegative due to our assumption that the corresponding transition rate (4.15) is always nonnegative.

Hence, let us consider the last term on the r.h.s. of eq. (5.15). Upon substituting expression (4.35) for $C_{\text{mem}}^{(3)}$ and shifting the integration variable $p \rightarrow p - p'/2$ in this term, we arrive at

$$\begin{aligned} \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{mem}}^{(3)} &= \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left[\tilde{L}^{+-}(X; p + \frac{p'}{2}, p - \frac{p'}{2}) - \tilde{L}^{-+}(X; p + \frac{p'}{2}, p - \frac{p'}{2}) \right] \\ &\times \left\{ L^{+-}(X, p'), L^{-+}(X, p') \right\}_{p', X} \ln \frac{\tilde{F}(X, p - p'/2)}{F(X, p - p'/2)}. \end{aligned} \quad (5.18)$$

By making the half-sum of eq. (5.18) with that with replaced $p' \rightarrow -p'$, we arrive at the symmetric form of this equation

$$\begin{aligned} \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{mem}}^{(3)} &= \frac{i}{4} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left[\tilde{L}^{-+}(X; p + \frac{p'}{2}, p - \frac{p'}{2}) - \tilde{L}^{+-}(X; p + \frac{p'}{2}, p - \frac{p'}{2}) \right] \\ &\times \left\{ L^{-+}(X, p'), L^{+-}(X, p') \right\}_{p', X} \ln \frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)}, \end{aligned} \quad (5.19)$$

which after simple algebraic transformations can be decomposed into two terms

$$\text{Tr} \int \frac{d^4 p}{(2\pi)^4} \ln \frac{\tilde{F}}{F} C_{\text{mem}}^{(3)} = c_{\text{mem}} + \delta c_{\text{mem}}, \quad (5.20)$$

with

$$\begin{aligned} c_{\text{mem}}(X) &= \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left\{ L^{-+}(X, p'), L^{+-}(X, p') \tilde{L}^{-+} \left(X; p + \frac{p'}{2}, p - \frac{p'}{2} \right) \right. \\ &\times \left. \left[\ln \frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)} - 1 \right] \right\}_{p', X}, \end{aligned} \quad (5.21)$$

$$\begin{aligned} \delta c_{\text{mem}}(X) &= \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left[\left\{ L^{-+}(X, p'), L^{+-}(X, p') \right\}_{p', X} \tilde{L}^{-+} \left(X; p + \frac{p'}{2}, p - \frac{p'}{2} \right) \right. \\ &- \left. \left\{ L^{-+}(X, p'), \tilde{L}^{-+} \left(X; p + \frac{p'}{2}, p - \frac{p'}{2} \right) \ln \frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)} \right\}_{p', X} \right. \\ &\times \left. L^{+-}(X, p') \right]. \end{aligned} \quad (5.22)$$

Using partial integration, we have subdivided the quantity of eq. (5.20) in such a way that the first term c_{mem} takes the form of the full divergence, and thus defines the non-Markovian contribution to the entropy flow (cf. eq. (5.16))

$$s_{\text{mem}}^\nu(X) = -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \tilde{L}^{-+} \left(X; p + \frac{p'}{2}, p - \frac{p'}{2} \right) L^{+-}(X, p')$$

$$\times \left[\ln \frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)} - 1 \right] \frac{\partial L^{-+}(X, p')}{\partial p'_\nu}. \quad (5.23)$$

This c_{mem} -term remains nonzero even in the local thermal equilibrium, which is defined by the former equilibrium relations (B.1)–(B.4) but with temperature $T(X)$, 4-velocity $U^\nu(X)$ and chemical potential $\mu(X)$ depending on the coordinates. On the contrary, as we show below, the second term δc_{mem} vanishes in the limit of local thermal equilibrium.

In local thermal equilibrium the Kubo-Martin-Schwinger condition (B.1) provides the following relations

$$\left(\frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)} \right)_{\text{loc.eq.}} = \left(\frac{L^{+-}(X; p')}{L^{-+}(X, p')} \right)_{\text{loc.eq.}} = \exp \left(\frac{p'_\nu U^\nu(X)}{T(X)} \right), \quad (5.24)$$

which can also be derived from (B.2)–(B.5) proceeding from definitions of L^{ij} (4.26) and \tilde{L}^{ij} (4.27). Guided by (5.24) we write the \tilde{L}^{ij} ratio in the ln-term of (5.22) as

$$\frac{\tilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\tilde{L}^{-+}(X; p + p'/2, p - p'/2)} = \frac{L^{+-}(X; p')}{L^{-+}(X, p')} (1 + \xi), \quad (5.25)$$

where ξ is expected to be small within the validity range of the generalized kinetic equation (3.27), i.e. $|\xi| \approx |f - \gamma|$. Substituting this into expression (5.22) for δc_{mem} , the $\ln(L^{-+}/L^{+-})$ -term in the second Poisson bracket cancels against the first term, since the p -integration converts the linear \tilde{L}^{ij} factor into L^{ij} factor. Thus one obtains

$$\begin{aligned} \delta c_{\text{mem}} &= -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left\{ L^{-+}, \tilde{L}^{-+} \ln(1 + \xi) \right\}_{p', X} L^{+-} \\ &\simeq \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left\{ L^{-+}, \tilde{L}^{-+} \frac{1}{2} \xi^2 \right\}_{p', X} L^{+-}, \end{aligned} \quad (5.26)$$

where also the term linear in ξ exactly cancels out, since

$$\begin{aligned} &-\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left\{ L^{-+}, \tilde{L}^{-+} \xi \right\}_{p', X} L^{+-} \\ &= -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \left\{ L^{-+}, \left(\frac{L^{-+}}{L^{+-}} \tilde{L}^{+-} - \tilde{L}^{-+} \right) \right\}_{p', X} L^{+-} = 0. \end{aligned} \quad (5.27)$$

As above, the \tilde{L}^{ij} can be readily integrated over p to produce L^{ij} , revealing a cancellation of the terms in the Poisson bracket in (5.27). Thus, δc_{mem} is not only zero in local equilibrium ($\xi = 0$). It is of the second-order in the small parameter $|\xi| \approx |f - \gamma|$ times gradients and therefore negligible compared to c_{mem} in (5.21) which is of linear order in $|\xi| \approx |f - \gamma|$ times gradients. Thus within the validity of the generalized kinetic equation (3.27) $s_{\text{mem}}^\nu(X)$

as given in (5.23) represents the appropriate non-Markovian memory correction to the entropy flow. In local equilibrium expression this term can be further simplified to

$$(s_{\text{mem}}^\nu)_{\text{eq}} = \frac{i}{2} \int \frac{d^4 p'}{(2\pi)^4} L^{-+}(X; p') L^{+-}(X, p') \left[\ln \frac{L^{+-}(X, p')}{L^{-+}(X, p')} - 1 \right] \frac{\partial L^{-+}(X, p')}{\partial p'_\nu} \quad (5.28)$$

by means of relation (5.24).

6 Thermodynamic Limit of Entropy

6.1 Thermodynamic Entropy

In the Matsubara technique, the thermodynamic potential Ω (see, e.g., ref. [4]) is a functional of Matsubara Green's functions $G(\varepsilon_n, \mathbf{p})$. Its important property is that it is stationary under variations of $G(\varepsilon_n, \mathbf{p})$ at fixed free Matsubara Green's function $G^0(\varepsilon_n, \mathbf{p})$

$$\left(\frac{\delta \Omega}{\delta G(\varepsilon_n, \mathbf{p})} \right)_{G^0} = 0. \quad (6.1)$$

Since $G(\varepsilon_n, \mathbf{p})$ has the spectral representation

$$G(\varepsilon_n, \mathbf{p}) = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi} \frac{A(\varepsilon', \mathbf{p})}{\varepsilon_n - \varepsilon'}, \quad (6.2)$$

property (6.1) implies that Ω is stationary under variations of the spectral density $A(\varepsilon, \mathbf{p})$, provided the Matsubara frequencies ε_n , over which the summation runs, are not changed and, thus, the free Matsubara Green's function $G^0(\varepsilon_n, \mathbf{p})$ is unaltered.

In the standard way, cf. (B.8)–(B.10), the ε_n -sum is converted into an energy integral over the distribution functions $n(\varepsilon)$, cf. (B.4), and thermodynamic potential expressed in terms of the real-time Green's functions and self-energies becomes [15] (cf. eqs. (B.2) and (B.3)),

$$\Omega = \text{Tr} \int d^3x \frac{d^4p}{(2\pi)^4} n(\varepsilon) \left(-2\text{Im} \ln [-G^R(p_0 + i0, \mathbf{p})] - \text{Re}G^R \Gamma - A\text{Re}\Sigma^R \right) + \Phi_T \quad (6.3)$$

where $\varepsilon = p_0 - \mu$ in the rest frame of the equilibrated system, and Φ_T is represented by the same set of closed diagrams as Φ . The thermodynamic entropy density is defined as

$$S = -(\partial \Omega / \partial T)_\mu / V, \quad (6.4)$$

i.e. as the temperature derivative at the fixed chemical potential μ with V being the system's volume. Due to the stationarity property of Ω , the only T -dependences to be taken into account in calculating the entropy are the explicit dependence of the discrete Matsubara frequencies on T or in the occupations $n(\varepsilon)$ in the integral formulation (6.3). From eqs (6.3) and (6.4) one finds

$$S = -\text{Tr} \int \frac{d^4 p}{(2\pi)^4} \frac{\partial n(\varepsilon)}{\partial T} \left[-2\text{Im} \ln [-G^R(p_0 + i0, \mathbf{p})] - \text{Re}G^R \Gamma - A\text{Re}\Sigma^R \right] - \frac{\partial \Phi_T}{\partial T} \quad (6.5)$$

for the entropy density. With the help of the identity (B.6) we replace $\partial n(\varepsilon)/\partial T$ in eq. (6.5) and then perform partial integration over p_0 . Thus, we obtain

$$S = S_{\text{loc}} + S_{\text{mem}}, \quad \text{where} \quad (6.6)$$

$$S_{\text{loc}} = -\text{Tr} \int \frac{d^4 p}{(2\pi)^4} \sigma(\varepsilon) \frac{\partial}{\partial p_0} \left[-2\text{Im} \ln [-G^R(p_0 + i0, \mathbf{p})] - \text{Re}G^R \Gamma \right], \quad (6.7)$$

$$S_{\text{mem}} = -\frac{\partial \Phi}{\partial T} + \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \sigma(\varepsilon) \frac{\partial (A\text{Re}\Sigma^R)}{\partial p_0}, \quad (6.8)$$

$$\sigma(\varepsilon) = \mp(1 \mp n(\varepsilon)) \ln(1 \mp n(\varepsilon)) - n(\varepsilon) \ln n(\varepsilon). \quad (6.9)$$

We have used subscripts “local” and “memory” to denote these two different contributions, because below we demonstrate that they are indeed associated with local (Markovian) s_{loc}^0 , cf. eq. (5.5), and memory (non-Markovian) s_{mem}^0 , cf. eq. (5.23), parts of the kinetic entropy. Taking derivatives in eq. (6.7), we readily get

$$S_{\text{loc}} = \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \sigma(\varepsilon) A_s^0(p) \quad (6.10)$$

with A_s^0 defined in eq. (5.7). Thus, S_{loc} , indeed, coincides with 0-component of the kinetic entropy flow (5.6).

In order to clarify the meaning of the values S_{loc} and S_{mem} we first inspect the quasiparticle limit, in which the spectral function is reduced to the delta-function, cf. eq. (3.33). In this limit, the value (6.10) for S_{loc} is reduced to

$$S_{\text{loc}}^{\text{qp}} = \text{Tr} \int \frac{d^3 p}{(2\pi)^3} \sigma(\varepsilon(\mathbf{p}) - \mu), \quad (6.11)$$

where the quasiparticle energy $\varepsilon(\mathbf{p})$ is determined by solution of dispersion equation (3.34). The full $S_{\text{loc}}^{\text{qp}}$ is just the sum of single-particle contributions, as if we deal with a noninteracting ideal gas of quasiparticles. This is a standard picture in the quasiparticle approximation. Corrections to $S_{\text{loc}}^{\text{qp}}$, resulting from S_{loc} , are of higher order in the width Γ . At the same time, S_{mem} provides corrections to $S_{\text{loc}}^{\text{qp}}$ even in the zero-order in Γ , which are associated with real rescatterings of on-mass-shell quasiparticles. This fact was demonstrated by

Carneiro and Pethick in ref. [15]. For the Fermi liquid, they showed that the first diagram for Φ that contributes to S_{mem} is the triangle one $\Phi^{(3)}$ of eq. (4.12). Contributions of $\Phi^{(1)}$ and $\Phi^{(2)}$ are zero. In the quasiparticle approximation, the contribution of $\Phi^{(3)}$ is (cf. eqs (45) and (72) of ref. [15])

$$S_{\text{mem}}^{\text{qp}} = \frac{1}{12} \int \frac{d^4 p}{(2\pi)^4} (\Gamma_b^{\text{qp}}(p))^3 \frac{\partial n_b(p_0)}{\partial T}, \quad (6.12)$$

where n_b is the thermal occupation number of artificial boson, given by eq. (B.4), and the bosonic width Γ_b is defined as

$$\Gamma_b(p) = V_0 d \int \frac{d^4 p'}{(2\pi)^4} \left[n_f \left(\varepsilon' - \frac{1}{2}\omega \right) n_f \left(\varepsilon' + \frac{1}{2}\omega \right) \right] A_f \left(p' - \frac{1}{2}p \right) A_f \left(p' + \frac{1}{2}p \right) \quad (6.13)$$

with $\varepsilon' = p'_0 - \mu$ and $\omega = p_0$, n_f stands for thermal fermion occupations, cf. eq. (B.4). As above, V_0 stands for the strength of the two-body potential. To get the quasiparticle approximation to this width (Γ_b^{qp}), we should replace the exact spectral function A_f in eq. (6.13) by its quasiparticle approximation A_f^{qp} defined by eq. (3.33). Note that $S_{\text{mem}}^{\text{qp}}$ is expressed in terms of bosonic quantities n_b and Γ_b , although we have started initially with the purely fermionic system with potential interaction. This fact provides a link to the thermodynamic calculation of Riedel [14], where the correction to the standard quasiparticle entropy of a Fermi liquid is presented in the form of the effective bosonic contribution. We remind that Riedel [14] summed all the series of ring diagrams rather than considered only three first of them as in eq. (4.12). At low temperatures, $S_{\text{mem}}^{\text{qp}} \sim T^3 \ln T$ [15], i.e. it gives the leading correction to the standard quasiparticle entropy. This is the famous correction to the specific heat of liquid ${}^3\text{He}$ [16,14,15]. Since this correction is quite comparable (numerically) to the leading term in the specific heat ($\sim T$), we may claim that the liquid ${}^3\text{He}$ is a liquid with quite strong memory from the point of view of kinetics.

Please also notice that different but an even more simple form of thermodynamic entropy including memory corrections follows from eq. (3.46) if one uses thermodynamic relation $E + PV - \mu N = TSV$:

$$TS = \text{Tr} \int \frac{d^4 p}{(2\pi)^4} An(\varepsilon) \left[\varepsilon - \frac{2}{3} \varepsilon_p^0 \right]. \quad (6.14)$$

6.2 Non-Markovian Entropy in Equilibrium

We now evaluate the memory correction $(s_{\text{mem}}^\nu)_{\text{eq}}$ to the kinetic entropy flow (see eq. (5.28)) in thermal equilibrium. Proceeding from the definition of L^{ij} (4.26), as well as from equilibrium relations (B.2)–(B.4), and identity (B.5), we present L^{ij} in the form

$$L^{-+}(p) = i n_b(\omega) \Gamma_b(p), \quad L^{+-}(p) = i [1 + n_b(\omega)] \Gamma_b(p), \quad (6.15)$$

where $\omega = p_\nu U^\nu$, Γ_b is defined by eq. (6.13), and n_b is the bosonic occupation number. Now, $(s_{\text{mem}}^\nu)_{\text{eq}}$ of eq. (5.28) takes the form

$$(s_{\text{mem}}^\nu)_{\text{eq}} = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} n_b(\omega) [1 + n_b(\omega)] \Gamma_b^2(p) \left(\frac{\omega}{T} - 1\right) \frac{\partial}{\partial p_\nu} [n_b(\omega) \Gamma_b(p)] \quad (6.16)$$

$$\begin{aligned} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} n_b(1 + n_b) \left(\frac{\omega}{T} - 1\right) \left(\Gamma_b^3 \frac{\partial n_b}{\partial p_\nu} + \frac{1}{3} \frac{\partial \Gamma_b^3}{\partial p_\nu} n_b\right) \\ &= \frac{1}{6} U^\nu \int \frac{d^4 p}{(2\pi)^4} \Gamma_b^3 \left[n_b^2 (1 + n_b) \frac{\omega}{T^2}\right]. \end{aligned} \quad (6.17)$$

The last line is obtained with the help of the partial integration, explicitly taking derivatives of bosonic occupations and using their property $d n_b / d \omega = -n_b(1 + n_b)/T$. Now, we change the integration variable $p \rightarrow -p$ in the last line of eq. (6.17) and use the parity properties of the relevant quantities

$$\Gamma_b(-\omega) = -\Gamma_b(\omega), \quad n_b(-\omega) = -[1 + n_b(\omega)], \quad (6.18)$$

and arrive at

$$(s_{\text{mem}}^\nu)_{\text{eq}} = -\frac{1}{6} U^\nu \int \frac{d^4 p}{(2\pi)^4} \Gamma_b^3 \left[n_b (1 + n_b)^2 \frac{\omega}{T^2}\right]. \quad (6.19)$$

Taking the half-sum of the r.h.s. in (6.19) and the last line of eq. (6.17) and using the identity (B.7) we finally arrive at

$$(s_{\text{mem}}^\nu)_{\text{eq}} = \frac{1}{12} U^\nu \int \frac{d^4 p}{(2\pi)^4} \Gamma_b^3 \frac{\partial n_b}{\partial T}. \quad (6.20)$$

The 0-component of (6.20) precisely coincides with the thermodynamic quantity $S_{\text{mem}}^{\text{QP}}$ given by eq. (6.12) in the quasiparticle approximation, provided we consider it in the rest frame of the matter, i.e. at $U^\nu = \{1, \mathbf{0}\}$. This fact again justifies the name “memory” for the thermodynamic quantity S_{mem} .

Thus, we have demonstrated that our kinetic entropy, including memory contributions, coincides with the thermodynamic entropy in thermal equilibrium.

7 Conclusion and Prospects

Generalized kinetic equations have been derived from the Kadanoff-Baym equations by means of the gradient approximation. Thereby the usual restriction to small mass widths

(mass-shell condition) for the particles involved is avoided. The equations are valid, if all phase-space distribution functions vary slowly across the space-time region. Since the mass-shell relation between energy and momentum is lost for unstable or virtual particles, the generalized kinetic equation is formulated in terms of a distribution function depending on energy and spatial momentum. Mass-shell equation loses its normal sense in context of quasiparticle approximation and rather becomes equivalent to the generalized kinetic equation. Alongside a local retarded equation has to be solved which provides the dynamical information about the spectral functions of the particles. Besides the usual drift and collision terms present in any transport equation, like in Landau's Fermi liquid theory [24,4,16], a genuine width and a fluctuation-dependent term appear in the generalized transport equation. The latter term is normally dropped along with the quasiparticle approximation which we avoided. This extra term gives rise to backflow contribution in the current and produces width and fluctuation dependent contributions to the energy momentum tensor pertaining to this transport scheme. It is indeed essential in order to preserve the conservation laws in the case of broad damping widths.

We suggest to follow Baym's Φ -derivable principle to consistently construct all terms for both, the kinetic and the retarded equation. It has a couple of important advantages, cf. ref. [2]. First, it leads to closed, i.e. self-consistent equations which can be closed at any order or loop level of the diagrams of Φ , this way defining an effective theory. We showed in ref. [8] that the original properties of Φ are also valid for genuine nonequilibrium systems described within the real-time formalism, namely that the so constructed approximation is conserving and at the same time thermodynamically consistent. In fact, for this to hold Baym showed that the Φ -derivable scheme is not only sufficient but even necessary! The Φ -derivable energy momentum tensor has explicitly been constructed for the contour Dyson equation in ref. [8]. Here we showed that the conserving properties also hold for the here derived generalized kinetic equations. As further shown in ref. [8] the scheme can be easily extended to include classical boson fields dynamically, such as mean fields or condensates. The latter permit to include soft modes in terms of these classical fields, much in the spirit of the kinetic picture [34,35,17] of hard loop approximations [36].

The structure of the collision term was studied by means of representing the Φ functional in terms of “ $-+$ ” and “ $+-$ ” Green's functions which represent the Wigner phase-space densities [17]. The advantage of the “ $-+$ ” and “ $+-$ ” representation is that it leads to a natural decomposition of the collision term into multi-particle processes with Feynman transition amplitudes which determine the partial rates. Furthermore, it has been discussed that sometimes it appears advantageous to account for such memory effects by including new “artificial” particles (e.g. bosonization of particle-hole excitations) which then lead to local collision terms.

We also addressed the question whether a closed nonequilibrium system approaches the thermodynamic equilibrium during its evolution. Investigating the structure of the collision term in the Φ -derivable scheme we obtained definite expressions for the local (Markovian) entropy flow and were able to explicitly demonstrate the H -theorem for some of the common choices of Φ approximations. The expression for the local entropy flow holds

beyond the quasiparticle picture, and thus generalizes the well-known Boltzmann expression for the kinetic entropy. To demonstrate memory effects in the generalized kinetics, we considered a particular case of a system of fermions interacting via two-body zero-range potential. We calculated the memory (non-Markovian) contribution to the kinetic entropy, which merges the equilibrium limit with its famous correction to the specific heat of liquid ^3He [14–16]: $\sim T^3 \ln T$. Since this correction is quite comparable (numerically) to the leading term in the specific heat ($\sim T$), one may claim that the liquid ^3He is a liquid with quite strong memory effects from the point of view of kinetics.

The mass-width effects are important for description of various physical systems. As for immediate applications of the developed formalism, we see the description of wide resonances (such as ρ -meson, Δ -resonance, etc.) in nonequilibrium hadron matter produced in heavy-ion collisions. Since the widths of these resonances are of the order (or even larger) than the excitation energy in the system, a self-consistent treatment of these widths is required. Up to now, width effects were considered either within some simplified dynamics with phenomenological Landau–Migdal residual interaction [37,38] or within a simple Φ -derivable approximation at thermal equilibrium and in the dilute gas limit [31]. In particular, it was demonstrated in ref. [17], that the soft-photon production is sensitive to dynamical and width effects. Interplay between the width and in-medium population of ρ -mesonic states may also simulate a shift of the ρ -meson mass in the nuclear medium, and thus affects the production of di-leptons in relativistic heavy-ion collisions. It is of interest to study these effects within a dynamical approach, such as the scheme presented here.

Further applications concern relativistic plasmas, such as QCD and QED plasmas. The plasma of deconfined quarks and gluons was present in the early Universe, it may exist in cores of massive neutron stars, and may also be produced in laboratory in ultra-relativistic nucleus–nucleus collisions. All these systems need a proper treatment of particle transport. Perturbative description of soft-quanta propagation suffers of infrared divergences and one needs a systematic study of the particle mass-width effects in order to treat them, cf. ref. [17]. A thermodynamic Φ -derivable approximation for hot relativistic QED plasmas—a gas of electrons and positrons in a thermal bath of photons—was recently considered by Vanderheyden and Baym [39]. Their treatment may be also applied to the high-temperature super-conductors and the fractional quantum Hall effect [40,41]. Our approach allows for a natural generalization of such a Φ -derivable schemes to the dynamical case.

Another application, as we see it, concerns the description of the neutrino transport in supernovas and hot neutron stars during first few minutes of their evolution. At an initial stage, neutrinos typically of thermal energy, produced outside (in the mantel) and inside the neutron-star core, are trapped within the regions of production. However, coherent effects in neutrino production and their rescattering on nucleons [17] reduce the opacity of the nuclear-medium and may allow for soft neutrinos to escape the core and contribute to the heating off the mantle. The extra energy transport may be sufficient to blow off the supernova’s mantle in the framework of the shock-reheating mechanism [42]. The

description of the neutrinos transport in the semi-transparent region should therefore be treated with the due account of mass-widths effects.

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APPENDICES

A Matrix Notation

In calculations that apply the Wigner transformations, it is necessary to decompose the full contour into its two branches—the *time-ordered* and *anti-time-ordered* branches. One then has to distinguish between the physical space-time coordinates x, \dots and the corresponding contour coordinates x^c which for a given x take two values $x^- = (x_\mu^-)$ and $x^+ = (x_\mu^+)$ ($\mu \in \{0, 1, 2, 3\}$) on the two branches of the contour (see figure 1). Closed real-time contour integrations can then be decomposed as

$$\int_C dx^c \dots = \int_{t_0}^\infty dx^- \dots + \int_{-\infty}^{t_0} dx^+ \dots = \int_{t_0}^\infty dx^- \dots - \int_{t_0}^\infty dx^+ \dots, \quad (\text{A.1})$$

where only the time limits are explicitly given. The extra minus sign of the anti-time-ordered branch can conveniently be formulated by a $\{-+\}$ "metric" with the metric tensor in $\{-+\}$ indices

$$(\sigma^{ij}) = (\sigma_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.2})$$

which provides a proper matrix algebra for multi-point functions on the contour with "co"- and "contra"-contour values. Thus, for any two-point function F , the contour values are defined as

$$\begin{aligned}
F^{ij}(x, y) &:= F(x^i, y^j), \quad i, j \in \{-, +\}, \quad \text{with} \\
F_i{}^j(x, y) &:= \sigma_{ik} F^{kj}(x, y), \quad F_j^i(x, y) := F^{ik}(x, y) \sigma_{ki} \\
F_{ij}(x, y) &:= \sigma_{ik} \sigma_{jl} F^{kl}(x, y), \quad \sigma_i^k = \delta_{ik}
\end{aligned} \tag{A.3}$$

on the different branches of the contour. Here summation over repeated indices is implied. Then contour folding of contour two-point functions, e.g. in Dyson equations, simply becomes

$$H(x^i, y^k) = H^{ik}(x, y) = \int_C dz^C F(x^i, z^C) G(z^C, y^k) = \int dz F_j^i(x, z) G^{jk}(z, y) \tag{A.4}$$

in the matrix notation.

For any multi-point function the external point x_{max} , which has the largest physical time, can be placed on either branch of the contour without changing the value, since the contour-time evolution from x_{max}^- to x_{max}^+ provides unity. Therefore, one-point functions have the same value on both sides on the contour.

Due to the change of operator ordering, genuine multi-point functions are, in general, discontinuous, when two contour coordinates become identical. In particular, two-point functions like $iF(x, y) = \langle T_C \hat{A}(x) \hat{B}(y) \rangle$ become

$$iF(x, y) = \begin{pmatrix} iF^{--}(x, y) & iF^{-+}(x, y) \\ iF^{+-}(x, y) & iF^{++}(x, y) \end{pmatrix} = \begin{pmatrix} \langle T \hat{A}(x) \hat{B}(y) \rangle & \mp \langle \hat{B}(y) \hat{A}(x) \rangle \\ \langle \hat{A}(x) \hat{B}(y) \rangle & \langle T^{-1} \hat{A}(x) \hat{B}(y) \rangle \end{pmatrix}, \tag{A.5}$$

where T and T^{-1} are the usual time and anti-time ordering operators. Since there are altogether only two possible orderings of the two operators, in fact given by the Wightman functions F^{-+} and F^{+-} , which are both continuous, not all four components of F are independent. Eq. (A.5) implies the following relations between nonequilibrium and usual retarded and advanced functions

$$\begin{aligned}
F^R(x, y) &= F^{--}(x, y) - F^{-+}(x, y) = F^{+-}(x, y) - F^{++}(x, y) \\
&:= \Theta(x_0 - y_0) (F^{+-}(x, y) - F^{-+}(x, y)), \\
F^A(x, y) &= F^{--}(x, y) - F^{+-}(x, y) = F^{-+}(x, y) - F^{++}(x, y) \\
&:= -\Theta(y_0 - x_0) (F^{+-}(x, y) - F^{-+}(x, y)),
\end{aligned} \tag{A.6}$$

where $\Theta(x_0 - y_0)$ is the step function of the time difference. The rules for the co-contour functions F_{--} etc. follow from eq. (A.3).

For such two point functions complex conjugation implies

$$\begin{aligned}
(\mathrm{i}F^{-+}(x, y))^* &= \mathrm{i}F^{-+}(y, x) \Rightarrow \mathrm{i}F^{-+}(X, p) = \text{real}, \\
(\mathrm{i}F^{+-}(x, y))^* &= \mathrm{i}F^{+-}(y, x) \Rightarrow \mathrm{i}F^{+-}(X, p) = \text{real}, \\
(\mathrm{i}F^{--}(x, y))^* &= \mathrm{i}F^{++}(y, x) \Rightarrow (\mathrm{i}F^{--}(X, p))^* = \mathrm{i}F^{++}(X, p), \\
(F^R(x, y))^* &= F^A(y, x) \Rightarrow (F^R(X, p))^* = F^A(X, p),
\end{aligned} \tag{A.7}$$

where the right parts specify the corresponding properties in the Wigner representation. Diagrammatically these rules imply the simultaneous swapping of all + vertices into - vertices and vice versa together with reversing the line arrow-sense of all propagator lines in the diagram.

In components the determination of the self-energy from the functional variation of Φ (cf. eq. (2.9)) reads

$$-\mathrm{i}\Sigma_{ik}(x, y) = \mp \frac{\delta\mathrm{i}\Phi}{\delta\mathrm{i}G^{ki}(y, x)} \Rightarrow -\mathrm{i}\Sigma_{ik}(X, p) = \mp \frac{\delta\mathrm{i}\Phi}{\delta\mathrm{i}G^{ki}(X, p)}, \quad i, k \in \{-+\} \tag{A.8}$$

the right expression given in the Wigner representation. Note that the variation over a $\{+-\}$ -“contra-variant” Green’s function G^{ki} produces a $\{+-\}$ -“covariant” self-energy Σ_{ik} , cf. (A.3). This occurs due to the same reason discussed above, cf. eq. (A.1), when dealing with $\{+-\}$ matrix notation with integrations over physical times, rather than contour times. The extra minus signs occurring for the anti-time ordered branches are precisely taken into account by the “covariant” notation (A.3).

B Equilibrium Relations

For completeness of the thermodynamic consideration, we explicitly present here equilibrium relations between quantities on the real-time contour. Basically, they follow from the Kubo–Martin–Schwinger condition [43]

$$G^{-+}(p) = \mp G^{+-}(p)e^{-\varepsilon/T}, \quad \Sigma^{-+}(p) = \mp \Sigma^{+-}(p)e^{-\varepsilon/T}, \tag{B.1}$$

where $\varepsilon = p_\nu U^\nu - \mu$ with U^ν and μ being a global 4-velocity of the system and a chemical potential related to the charge, respectively. All the Green’s functions can be expressed through the retarded and advanced Green’s functions

$$G^{ik}(p) = \begin{pmatrix} [1 \mp n(\varepsilon)] G^R(p) \pm n(\varepsilon) G^A(p) & \pm \mathrm{i}n(\varepsilon) A(p) \\ -\mathrm{i}[1 \mp n(\varepsilon)] A(p) & -[1 \mp n(\varepsilon)] G^A(p) \mp n(\varepsilon) G^R(p) \end{pmatrix}, \tag{B.2}$$

i, k mean + or $-$, and the self-energies take a similar form

$$\Sigma_{ik}(p) = \begin{pmatrix} \Sigma^R(p) \pm i n(\varepsilon) \Gamma(p) & \mp i n(\varepsilon) \Gamma(p) \\ i [1 \mp n(\varepsilon)] \Gamma(p) & -\Sigma^A(p) \pm i n(\varepsilon) \Gamma(p) \end{pmatrix}. \quad (\text{B.3})$$

Here

$$n(\varepsilon) = [\exp(\varepsilon/T) \pm 1]^{-1} \quad (\text{B.4})$$

are thermal Fermi/Bose–Einstein occupations. They obey some useful relations between fermion n_f and boson n_b occupation numbers, like

$$n_{f,b}(\varepsilon + \omega/2) [1 \mp n_{f,b}(\varepsilon - \omega/2)] = [n_{f,b}(\varepsilon - \omega/2) - n_{f,b}(\varepsilon + \omega/2)] n_b(\omega), \quad (\text{B.5})$$

or derivatives with respect to T

$$\frac{\partial n(\varepsilon)}{\partial T} = -\frac{\partial \sigma(\varepsilon)}{\partial \varepsilon}, \quad \sigma(\varepsilon) = \mp[1 \mp n(\varepsilon)] \ln[1 \mp n(\varepsilon)] - n(\varepsilon) \ln n(\varepsilon), \quad (\text{B.6})$$

$$\frac{\partial n(\varepsilon)}{\partial T} = -\frac{\omega}{T^2} n_b(1 \mp n_b). \quad (\text{B.7})$$

The link between the Matsubara technique and the real-time formulation used here can be provided by extending the real time contour by an imaginary tail going to $-i\beta$, $\beta = 1/T$, this way defining the equilibrium density operator $\exp(-\beta \hat{H})$. Thus, the link is provided by considering analytic expressions like the "contour trace" of two-point functions $iF_{eq}(x, y) = \langle \mathcal{T}_C \hat{A}(x) \hat{B}(y) \rangle$

$$\begin{aligned} \int_{\mathcal{C}\{eq\}} F_{eq}(t, t+0) dt &= \int_0^{-i\beta} F_{eq}^{--}(t, t) dt = \mp i\beta \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n(\varepsilon) (F_{eq}^A(\omega) - F_{eq}^R(\omega)) \\ &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{F_{eq}^S(\omega)}{i\varepsilon_m + \mu_{AB} - \omega} = \sum_{m=-\infty}^{\infty} F^{Matsubara}(i\varepsilon_m). \end{aligned} \quad (\text{B.8})$$

Here the contour time $t+0$ is placed infinitesimally behind t on the contour in order to specify a *fixed* operator ordering of the two external operators of F , μ_{AB} is the chemical potential associated to $\langle \hat{A}(x) \hat{B}(y) \rangle$, and $\mu_{AB} = -\mu_{BA}$. The step towards the discrete Matsubara sum is provided by standard residue technique, cf. ref. [44], fig. 25.4 ff. The sum runs over the Matsubara energies

$$\varepsilon_m = \begin{cases} (2m+1)\pi T & \text{for fermions} \\ 2m\pi T & \text{for bosons.} \end{cases} \quad (\text{B.9})$$

Thereby, the Matsubara form of the two-point function F_{eq} has the spectral representation

$$F^{Matsubara}(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{F_{eq}^S(\omega)}{z + \mu_{AB} - \omega} = \begin{cases} F_{eq}^R(z + \mu_{AB}) & \text{for } \text{Im } z > 0 \\ F_{eq}^A(z + \mu_{AB}) & \text{for } \text{Im } z < 0 \end{cases} \quad (\text{B.10})$$

in terms of the *real time contour* spectral function $F_{eq}^S = -2\text{Im}F_{eq}^R$.

C Diagram rules

For relativistic theories with local vertices the diagrammatic rules on the contour are identical to the standard Feynman rules except that all time integrations are to be replaced by contour integrations. The diagrams contributing to Φ are calculated as diagrams with one external point, namely the interaction part of the Lagrangian $\langle \hat{\mathcal{L}}^{\text{int}}(x) \rangle$, then contour integrated over x and weighted with $1/n_\lambda$, where n_λ counts the number vertices in the diagram. These diagrams have to be two-particle irreducible with all lines representing full propagators. For details about the corresponding diagrammatic rules on the contour see ref. [8].

The rules for nonrelativistic two-body interactions are also naturally extended to the contour \mathcal{C} with

$$\hat{H}^{\text{int}}(t_1) = \frac{1}{2} \int d^3x_1 \int_{\mathcal{C}} d^4x_2 \hat{\Psi}^\dagger(x_1) \hat{\Psi}^\dagger(x_2) V(x_1 - x_2) \hat{\Psi}(x_2) \hat{\Psi}(x_1) \quad (\text{C.1})$$

$$\text{with } V(x_1 - x_2) = U(\mathbf{x}_1 - \mathbf{x}_2) \delta_{\mathcal{C}}(t_1 - t_2) \quad (\text{C.2})$$

now defined for contour times t_1, t_2 . One has however to observe that for the instantaneous two-body interactions the diagrammatic elements are given by  $= -iV(x_1 - x_2)$, i.e. without the factor $1/2$ contained in (C.1). With respect to the topological rules, indeed, the interaction lines are treated like bosons in the relativistic theory with local vertices. To this extend we recall that the expectation value for $\langle \hat{H}^{\text{int}}(t) \rangle = \frac{1}{2} \times \{ \text{diagrams} \}$ has an explicit factor $\frac{1}{2}$ in front of the corresponding diagrams. Correspondingly, for the n_λ vertex counting for the diagrams of Φ each two-body interaction line counts as *two* vertices, cf. the rules given in ref. [4] for the thermodynamic potential in Matsubara formalism. However, the two-particle irreducibility of Φ -diagrams is defined with respect to cutting dynamical propagators only, i.e. leaving the V -interaction lines untouched. In $\{-+\}$ -matrix notation the above rules imply, cf. [21] sect. X,

$$\begin{aligned}
(+\text{---}\text{---})^* &= -\text{---}\text{---} = -iV(x_1 - x_2) \\
-\text{---}\text{---}^+ &= +\text{---}\text{---}^- = 0.
\end{aligned} \tag{C.3}$$

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